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ANTENNA THEORY

SOLUTIONS MANUAL TO ACCOMPANY

ANTENNA THEORY
ANALYSIS AND DESIGN

FOURTH EDITION

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WILEY

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Contents

Preface

This Solutions Manual consists of solutions for all the problems found in ***Antenna Theory: Analysis and Design*** (4th edition, 2016) at the end of Chapters 2–16. There are 699 (103 new) problems, most of them with multiple parts. The degree of difficulty and length varies. While certain solutions need special functions, found in graphical form in the appendices, others require the use of the computer program. These computer programs are placed on a password protected website. All of the computer programs, especially those at the end of Chapters 6, 9, 11, 13 and 14 have been developed to design, respectively, uniform and nonuniform arrays, impedance transformers, log-periodic arrays, horns and microstrip patch antennas. In some cases, the computer programs also perform analysis on the designs. The programs at the end of Chapters 2, 4, 5, 7, 8, 10, 12, 15 and 16 are primarily developed for analysis. The problems have been designed to test the student's grasp of this text's material and to apply the concepts to the analysis and design of many practical radiators. In this fourth edition, more emphasis has been placed on design. To accomplish this, equations, procedures, examples, graphs, end-of-the-chapter problems, and computer programs have been developed.

This manual has been prepared to assist the instructor in making homework and test assignments, and to provide one set of solutions for all of the problems. There maybe undoubtedly errors which have been overlooked. In addition, the solutions contained in this manual are not necessarily the simplest and/or the best. The author will, therefore, appreciate having errors brought to his attention and solicits alternate solutions to the problems.

This Solutions Manual for the fourth edition has been prepared from the manuals of the first, second and third editions and many other new problems provided by the author.

CHAPTER 2

Solution Manual

Exact

2.1. (a) $d\Omega = \sin\theta \, d\theta \, d\phi$

$$\Omega_A = \int_{45°}^{60°}\int_{30°}^{60°} d\Omega = \int_{\pi/4}^{\pi/3}\int_{\pi/6}^{\pi/3} \sin\,\theta \, d\theta \, d\phi$$

$$= (\phi)\Big|_{\pi/4}^{\pi/3}(-\cos\,\theta)\Big|_{\pi/6}^{\pi/3}$$

$$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right)(-0.5 + 0.866)$$

$$\Omega_A = \left(\frac{\pi}{12}\right)(0.366) = 0.09582 \text{ sterads}$$

$$\Omega_A = \begin{cases} 0.09582 \text{ sterads} \\ 0.09582\left(\frac{180}{\pi}\right)\left(\frac{180}{\pi}\right) = 314.5585 \text{ (degrees)}^2 \end{cases}$$

Approximate

$$\Omega_A \simeq \left(\frac{\pi}{3} - \frac{\pi}{4}\right)\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

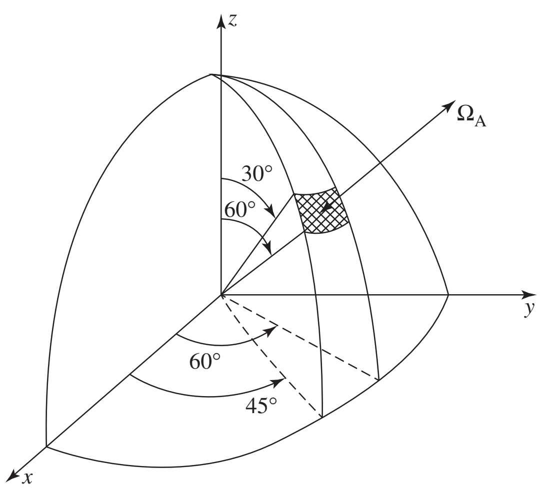
$$\simeq \left(\frac{\pi}{12}\right)\left(\frac{\pi}{6}\right) = \frac{\pi^2}{72}$$

$\Omega_A \simeq 0.13708$ sterads

$\Omega_A \simeq (60 - 45)(60 - 30)$

$\simeq 450$ (degrees)^2 or error of

$$\left(\frac{450 - 314.5585}{314.5585}\right) \times 100 = 43.06\%$$



Antenna Theory: Analysis and Design, Fourth Edition. Constantine A. Balanis.

Companion Website: www.wiley.com/go/antennatheory4e

(b) $$D_0 = \frac{4\pi}{\Omega_A(\text{sterads})} = \frac{4\pi}{0.09582} = 131.1456 \text{ (dimensionless)}$$
$$= 10\log_{10}(131.1456) = 21.1775 \text{ dB}$$

or

$$D_0 = \frac{4\pi\left(\frac{180}{\pi}\right)\left(\frac{180}{\pi}\right)}{\Omega_A(\text{degrees})^2} = 131.1456 \text{ (dimensionless)} = 21.1775 \text{ dB}$$

$$D_0 = \begin{cases} 131.1456 \text{ (dimensionless)} \\ 21.1775 \text{ (dB)} \end{cases}$$

2.2. $\underline{\mathcal{W}} = \underline{\mathcal{E}} \times \underline{\mathcal{H}} = \text{Re}\left[\underline{E}e^{j\omega t}\right] \times \text{Re}\left[\underline{H}e^{j\omega t}\right]$

Using the identity Re $\left[\underline{A}e^{j\omega t}\right] = \frac{1}{2}\left[\underline{E}e^{j\omega t} + \underline{E}^* e^{-j\omega t}\right]$

The instant Poynting vector can be written as

$$\begin{aligned}\underline{\mathcal{W}} &= \left\{\frac{1}{2}[\underline{E}e^{j\omega t} + \underline{E}^* e^{-j\omega t}]\right\} \times \left\{\frac{1}{2}[\underline{H}e^{j\omega t} + \underline{H}^* e^{-j\omega t}]\right\} \\ &= \frac{1}{2}\left\{\frac{1}{2}[\underline{E}\times\underline{H}^* + \underline{E}^*\times\underline{H}] + \frac{1}{2}[\underline{E}\times\underline{H}e^{j2\omega t} + \underline{E}^*\times\underline{H}^* e^{-j\omega t}]\right\} \\ &= \frac{1}{2}\left\{\frac{1}{2}[\underline{E}\times\underline{H}^* + (\underline{E}\times\underline{H}^*)^*] + \frac{1}{2}[\underline{E}\times\underline{H}e^{j2\omega t} + (\underline{E}\times\underline{H}e^{j\omega t})^*]\right\}\end{aligned}$$

Using the above identity again, but this time in reverse order, we can write that

$$\underline{\mathcal{W}} = \frac{1}{2}[\text{Re}(\underline{E}\times\underline{H}^*)] + \frac{1}{2}[\text{Re}(\underline{E}\times\underline{H}e^{j2\omega t})]$$

2.3. (a) $\underline{W}_{\text{rad}} = \frac{1}{2}\text{Re}[\underline{E}\times\underline{H}^*] = \frac{E^2}{2\eta}\hat{a}_r = \frac{5^2}{2(120\pi)}\hat{a}_r = 0.03315\hat{a}_r \text{ Watts/m}^2$

(b) $$\begin{aligned}P_{\text{rad}} &= \oint_s W_{\text{rad}}\, ds = \int_0^{2\pi}\int_0^{\pi}(0.03315)(r^2\sin\theta\, d\theta\, d\phi) \\ &= \int_0^{2\pi}\int_0^{\pi}(0.03315)(100)^2 \sin\theta\, d\theta\, d\phi \\ &= 2\pi(0.03315)(100)^2\int_0^{\pi}\sin\theta\, d\theta = 2\pi(0.03315)(100)^2\cdot(2) \\ &= 4165.75 \text{ Watts}\end{aligned}$$

2.4. (a) $U(\theta) = \cos\theta$

$$U(\theta_h) = 0.5 = \cos\theta_h \Rightarrow \theta_h = \cos^{-1}(0.5) = 60^\circ$$
$$\Rightarrow \Theta_h = 2(60^\circ) = 120^\circ = \frac{2\pi}{3} \text{ rads.}$$
$$U(\theta_n) = 0 = \cos\theta_n \Rightarrow \theta_n = \cos^{-1}(0) = 90^\circ$$
$$\Rightarrow \Theta_n = 2(90^\circ) = 180^\circ = \pi \text{ rads.}$$

(b) $U(\theta) = \cos^2\theta$

$$U(\theta_h) = 0.5 = \cos^2\theta_h \Rightarrow \theta_h = \cos^{-1}(0.5)^{1/2} = 45^\circ$$
$$\Rightarrow \Theta_h = 2(45) = 90^\circ = \pi/2 \text{ rads}$$
$$U(\theta_n) = 0 = \cos^2\theta_n \Rightarrow \theta_n = \cos^{-1}(0) = 90^\circ$$
$$\Rightarrow \Theta_n = 2(90^\circ) = 180^\circ = \pi \text{ rads}$$

(c) $U(\theta) = \cos(2\theta)$

$$U(\theta_h) = 0.5 = \cos(2\theta_h) \Rightarrow \theta_h = \frac{1}{2}\cos^{-1}(0.5) = 30^\circ$$
$$\Rightarrow \Theta_h = 2(30^\circ) = 60^\circ = \pi/3 \text{ rads}$$
$$U(\theta_n) = 0 = \cos(2\theta_n) \Rightarrow \theta_n = \frac{1}{2}\cos^{-1}(0) = 45^\circ$$
$$\Rightarrow \Theta_n = 2(45^\circ) = 90^\circ = \pi/2 \text{ rads}$$

(d) $U(\theta) = \cos^2(2\theta)$

$$U(\theta_h) = 0.5 = \cos^2(2\theta_h) \Rightarrow \theta_h = \frac{1}{2}\cos^{-1}(0.5)^{1/2} = 22.5^\circ$$
$$\Rightarrow \Theta_h = 2(22.5^\circ) = 45^\circ = \frac{\pi}{4} \text{ rads}$$
$$U(\theta_n) = 0 = \cos^2(2\theta_n) \Rightarrow \theta_n = \frac{1}{2}\cos^{-1}(0) = 45^\circ$$
$$\Rightarrow \Theta_n = 2(45^\circ) = 90^\circ = \pi/2 \text{ rads}$$

(e) $U(\theta) = \cos(3\theta)$

$$U(\theta_h) = \cos(3\theta_h) = 0.5 \Rightarrow \theta_h = \frac{1}{3}\cos^{-1}(0.5) = 20^\circ$$
$$\Rightarrow \Theta_h = 2(20^\circ) = 40^\circ = 0.698 \text{ rads}$$
$$U(\theta_n) = \cos(3\theta_n) = 0 \Rightarrow \theta_n = \frac{1}{3}\cos^{-1}(0) = 30^\circ$$
$$\Rightarrow \Theta_n = 2(30^\circ) = 60^\circ = \pi/3 \text{ rads}$$

(f) $U(\theta) = \cos^2(3\theta)$

$$U(\theta_h) = 0.5 = \cos^2(3\theta_h) \Rightarrow \theta_h = \frac{1}{3}\cos^{-1}(0.5)^{1/2} = 15^\circ$$
$$\Rightarrow \Theta_h = 2(15^\circ) = 30^\circ = \pi/6 \text{ rads}$$
$$U(\theta_n) = 0 = \cos^2(3\theta_n) \Rightarrow \theta_n = \frac{1}{3}\cos^{-1}(0) = 30^\circ$$
$$\Rightarrow \Theta_n = 2(30^\circ) = 60^\circ = \pi/3 \text{ rads}$$

2.5. Using the results of Problem 2.4 and a nonlinear solver to find the half power beamwidth of the radiation intensity represented by the transcentendal functions, we have that:

(a) $U(\theta) = \cos\theta\ \cos(2\theta) \Rightarrow \begin{cases} \text{HPBW} = 55.584^\circ \\ \text{FNBW} = 90^\circ \end{cases}$

(b) $U(\theta) = \cos^2\theta \ \cos^2(2\theta) \Rightarrow \begin{cases} \text{HPBW} = 40.985^\circ \\ \text{FNBW} = 90^\circ \end{cases}$

(c) $U = \cos\theta \ \cos(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 38.668^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

(d) $U = \cos^2\theta \ \cos^2(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 28.745^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

(e) $U = \cos(2\theta) \ \cos(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 34.942^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

(f) $U = \cos^2(2\theta) \ \cos^2(3\theta) \Rightarrow \begin{cases} \text{HPBW} = 25.583^\circ \\ \text{FNBW} = 60^\circ \end{cases}$

2.6. (a) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} = \dfrac{4\pi(200\times10^{-3})}{0.9(125.66\times10^{-3})} = 22.22 = 13.47 \text{ dB}$

$G_0 = \varepsilon_{cd} D_0 = 0.9(22.22) = 20 = 13.01 \text{ dB}$

(b) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} = \dfrac{4\pi(200\times10^{-3})}{(125.66\times10^{-3})} = 20 = 13.01 \text{ dB}$

$G_0 = \varepsilon_{cd} D_0 = 0.9\cdot(20) = 18 = 12.55 \text{ dB}$

2.7. $\boxed{U = B_0 \ \cos^2\theta}$

(a)
$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi/2} U \ \sin\theta \ d\theta = 2\pi B_0 \int_0^{\pi/2} \cos^2\theta \ \sin\theta \ d\theta$$
$$= 2\pi B_0 \int_0^{\pi/2} \cos^2 \ \theta \ d\,(-\cos\theta)$$
$$P_{\text{rad}} = -2\pi B_0 \frac{\cos^3\theta}{3}\bigg|_0^{\pi/2} = -2\pi B_0\left[\frac{-1}{3}\right] = \frac{2\pi}{3}B_0 = 10 \Rightarrow B_0 = \frac{15}{\pi}$$
$$U = \frac{15}{\pi}\cos^2\theta \ \Rightarrow W_{\text{rad}}\bigg|_{\max} = \frac{U}{r^2}\bigg|_{\max} = \frac{15}{\pi}\ \frac{\cos^2\theta}{r^2}\bigg|_{\max}$$
$$= \frac{15}{\pi(10^3)^2} = 4.7746\times10^{-6} \text{ Watts/m}^2 @ \ \theta = 0^\circ$$

$$W_{\text{rad}}\bigg|_{\max} = 4.7746\times10^{-6} \text{ Watts/m}^2 @ \ \theta = 0^\circ$$

(b)
$$\Omega_A \text{ (exact)} = \int_0^{2\pi}\int_0^{\pi} U_n \ \cos^2\theta \ \sin\theta \ d\theta \ d\phi$$
$$\Omega_A \text{ (exact)} = \frac{2\pi}{3} \text{ steradians} = 2.0944 \text{ sterads} = 6,875.51 \text{ (degrees)}^2$$
$$U = 0.5 = \cos^2\theta_h \Rightarrow \theta_h = \cos^{-1}(0.5)^{1/2} = 45^\circ$$
$$\Rightarrow \Theta_h = 2(45^\circ) = 90^\circ = \pi/2 \text{ rads}$$
$$\Omega_A\left(\frac{\text{Kraus'}}{\text{approx}}\right) = \Theta_h^2 = (\pi/2)^2 = \frac{\pi^2}{4} = 2.4674 \text{ sterads} = 8,099.997 \text{ (degrees)}^2$$

(c) $$D_0 \text{ (exact)} = \frac{4\pi}{\Omega_A(\text{exact})} = \frac{4\pi}{2\pi/3} = 6 = 7.782 \text{ dB}$$

$$D_0 \text{ (approx/Kraus')} = \frac{4\pi}{\Omega_A(\text{approx})} = \frac{4\pi}{\pi^2/4} = \frac{16}{\pi} = 5.093 = 7.0697$$

(d) G_0 Assuming lossless antenna ($P_{in} = P_{\text{rad}}$)

$$G_0 \text{ (exact)} = D_0 \text{ (exact)} = 6 = 7.782 \text{ dB}$$

$$G_0 \text{ (approx)} = D_0 \text{ (approx)} = 5.093 = 7.0697 \text{ dB}$$

$$\boxed{U = B_0 \cos^3 \theta}$$

(a) $$P_{\text{rad}} = -2\pi B_0 \left(-\frac{1}{4}\right) = \frac{\pi}{2} B_0 = 10 \Rightarrow B_0 = 20/\pi$$

$$W_{\text{rad}}\Big|_{\max} = \frac{20}{\pi}\frac{1}{\pi^2} = \frac{20}{\pi} \times 10^{-6} = 6.366 \times 10^{-6} \text{ Watts/m}^2$$

(b) Ω_A (exact) = $(\pi/2) = 1.5708$ sterads

$$U = 0.5 = \cos^3 \theta_h \Rightarrow \theta_h \ \cos^{-1}(0.5)^{1/3} = 37.467°$$

$$\Rightarrow \Theta_h = 2(37.467°) = 74.934° = 1.30785 \text{ rads}$$

$$\Omega_A \text{ (approx)} = (1.30785)^2 = 1.71 \text{ sterads}$$

(c) D_0 (exact) $= 4\pi/(\pi/2) = 8 = 9.031$ dB

$$D_0 \text{ (approx)} = \frac{4\pi}{1.71} = 7.347 = 8.66 \text{ dB}$$

(d) Assuming lossless antenna ⇒ Gain = Directivity (see part c)

2.8. $\underline{E}_a = \hat{a}_\theta E_a \sin^{1.5}\theta \frac{e^{-jkr}}{r} \Rightarrow U_n = (\sin^{1.5}\theta)^2 = \sin^3\theta$ **Normalized U_n**

(a) $$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}, U_{\max} = U_n|_{\max} = \sin^3\theta \Big|_{\substack{\max \\ \theta=\theta\max}} = 1,\ \theta_{\max} = 90°$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U_n \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi} \sin^3\theta \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi} \sin^4\theta \, d\theta$$

$$= 2\pi\left[-\frac{\sin^3\theta\cos\theta}{4}\Big|_0^{\pi} + \frac{3}{4}\int_0^{\pi} \sin^2\theta \, d\theta\right] = 2\pi\left[\frac{3}{4}\left(\frac{1}{2} - \frac{1}{4}\sin(2\theta)\right)_0^{\pi}\right]$$

$$= 2\pi\left[\frac{3}{4}\left(\frac{\pi}{2}\right)\right] = \frac{3\pi^2}{4}$$

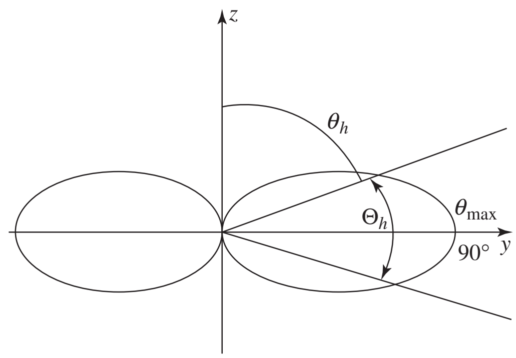
$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(1)}{3\pi^2/4} = \frac{16}{3\pi} = \boxed{1.698 = 2.298 \text{ dB}}$$

(b) $U_n = \sin^3\theta, U_{n\max} = 1, \theta_{\max} = 90°$

$$U_n\Big|_{\theta=\theta_h} = 0.5 = \sin^3\theta_h \Rightarrow \theta_h = \left[\sin^{-1}(0.5^4)^3\right] = \sin^{-1}(0.794) = 52.533°$$

$$\text{HPBW} = \Theta_h = 2(\theta_{\max} - \theta_h) = 2(90 - 52.533)$$

$$\boxed{\Theta_h = 2(37.467) = 74.934°}$$



(c) Because pattern is **omnidirectional**:

$$D_0(\text{McDonald}) = \frac{101}{\text{HPBW} - 0.0027(\text{HPBW})^2} = \frac{101}{74.934 - 0.0027(74.934)^2}$$

$$D_0(\text{McDonald}) = \frac{101}{74.934 - 15.161} = \frac{101}{59.773} = \boxed{1.690 = 10\log_{10}(1.690) = 2.278}$$

(d) Because pattern is **omnidirectional**:

$$D_0(\text{Pozar}) = -172.4 + 191\sqrt{0.818 + \frac{1}{\text{HPBW}}} = -172.4 + 191\sqrt{0.818 + \frac{1}{74.934}}$$

$$= -172.4 + 191(0.912) = -172.4 + 174.150 = 1.750$$

$$P_0(\text{Pozar}) = \boxed{1.750 = 10\log_{10}(1.750) = 2.431 \text{ dB}}$$

(e) Computer Program Directivity: $\boxed{D_0 = 1.693 = 2.2864 \text{ dB}}$

```
Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 180
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
-------------------
Radiated power (watts) = 7.4228
Directivity (dimensionless) = 1.6930
Directivity (dB) = 2.2864
```

2.9. $U(\theta, \phi) = \cos^n(\theta) \qquad 0 \le \theta \le \pi/2, 0 \le \phi \le 2\pi$

(a) $U_n(\theta_n, \phi) = 0.5 = \cos^n(5°) = [\cos(5°)]^n = (0.99619)^n$

$$0.5 = (0.99619)^n$$

$$\log_{10}(0.5) = \log[(0.99619)^n] = n\log_{10}(0.99619) = n(-0.00166)$$

$$-0.30103 = -0.00166n$$

$$\boxed{n = 181.34}$$

(b) $U(\theta,\phi) = \cos^{181.34}(\theta); U_{\max} = 1, \theta = 0°$

$$P_{\rm rad} = \int_0^{2\pi}\int_0^{\pi/2} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = 2\pi \int_0^{\pi/2} \cos^{181.34}(\theta)\sin\theta\, d\theta$$

$$= 2\pi\left[-\frac{\cos^{182.34}(\theta)}{182.34}\right]_0^{\pi/2} = \left[-0 + \frac{1}{182.34}\right]2\pi = \frac{2\pi}{182.34} = 0.03446$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\rm rad}} = \frac{4\pi(1)}{2\pi}(182.34) = 2(182.34) = 364.68$$

$$\boxed{D_0 = 364.68 = 25.62 \text{ dB}}$$

(c) *Kraus' Approximation* (2.27):

$$D_0 \simeq \frac{41,253}{\Theta_{1d}\Theta_{2d}} = \frac{41,253}{(10)(10)} = 412.53 = 26.15 \text{ dB}$$

$$\boxed{D_0 \simeq 412.53 = 26.15 \text{ dB}}$$

(d) *Tai & Pereira* (2.30b):

$$D_0 \simeq \frac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \frac{72{,}815}{2(10)^2} = \frac{72{,}815}{200} = 364.075 = 25.61 \text{ dB}$$

$$\boxed{D_0 \simeq 364.075 = 25.61 \text{ dB}}$$

2.10.

$$U(\theta,\phi) = \left\{\begin{matrix} 1 & 0° \le \theta \le 20° \\ 0.342\csc(\theta) & 20° \le \theta \le 60° \\ 0 & 60° \le \theta \le 180° \end{matrix}\right\} 0° \le \phi \le 360°$$

$$P_{\rm rad} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi$$

$$= 2\pi\left[\int_0^{20°}\sin\theta\, d\theta + \int_{20°}^{60°} 0.342\csc(\theta)\times\sin\theta\, d\theta\right]$$

$$= 2\pi\left\{-\cos\theta\Big|_0^{\pi/9} + 0.342\cdot\theta\Big|_{\pi/9}^{\pi/3}\right\}$$

$$= 2\pi\left\{\left[-\cos\left(\frac{\pi}{9}\right)+1\right] + 0.342\left(\frac{\pi}{3}-\frac{\pi}{9}\right)\right\}$$

$$= 2\pi\left\{[-0.93969+1] + 0.342\pi\left(\frac{2}{9}\right)\right\}$$

$$= 2\pi\{0.06031 + 0.23876\} = 1.87912$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\rm rad}} = \frac{4\pi(1)}{1.87912} = 6.68737 = 8.25255 \text{ dB}$$

2.11. (a) $D_0 \simeq \dfrac{41{,}253}{\Theta_{1d}\Theta_{2d}} = \dfrac{41{,}253}{30(35)} = 39.29 = 15.94 \text{ dB}$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0$$

(b) $D_0 \simeq \dfrac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \dfrac{72{,}815}{(30)^2 + (35)^2} = 34.27 = 15.35 \text{ dB}$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0$$

2.12. $D_0 = \dfrac{4\pi U_{max}}{P_{rad}}$

(a) $U = \sin\theta \sin\phi \quad \text{for} \quad 0 \le \theta \le \pi, 0 \le \phi \le \pi$

$U\big|_{max} = 1$ and it occurs when $\theta = \phi = \pi/2$.

$$P_{rad} = \int_0^{\pi} \int_0^{\pi} U \sin\theta \, d\theta \, d\phi = \int_0^{\pi} \sin\phi \, d\phi \int_0^{\pi} \sin^2\theta \, d\theta = 2\left(\frac{\pi}{2}\right) = \pi.$$

Thus $D_0 = \dfrac{4\pi(1)}{\pi} = 4 = 6.02 \text{ dB}$

The half-power beamwidths are equal to

$$\text{HPBW (az.)} = 2[90^\circ - \sin^{-1}(1/2)] = 2(90^\circ - 30^\circ) = 120^\circ$$
$$\text{HPBW (el.)} = 2[90^\circ - \sin^{-1}(1/2)] = 2(90^\circ - 30^\circ) = 120^\circ$$

In a similar manner, it can be shown that for the following:

(b) $U = \sin\theta \sin^2\phi \Rightarrow D_0 = 5.09 = 7.07 \text{ dB}$

$$\text{HPBW (el.)} = 120^\circ, \text{HPBW (az.)} = 90^\circ$$

(c) $U = \sin\theta \sin^3\phi \Rightarrow D_0 = 6 = 7.78 \text{ dB}$

$$\text{HPBW (el.)} = 120^\circ, \text{HPBW (az.)} = 74.93^\circ$$

(d) $U = \sin^2\theta \sin\phi \Rightarrow D_0 = 12\pi/8 = 4.71 = 6.73 \text{ dB}$

$$\text{HPBW (el.)} = 90^\circ, \text{HPBW (az.)} = 120^\circ$$

(e) $U = \sin^2\theta \sin^2\phi \Rightarrow D_0 = 6 = 7.78 \text{ dB}$

$$\text{HPBW (az.)} = \text{HPBW (el.)} = 90^\circ$$

(f) $U = \sin^2\theta \sin^3\phi \Rightarrow D_0 = 9\pi/4 = 7.07 = 8.49 \text{ dB}$

$$\text{HPBW (el.)} = 90^\circ, \text{HPBW (az.)} = 74.93^\circ$$

2.13. $U = \sin\theta \cos^2\phi, \quad 0 \le \theta \le 180^\circ, \quad 90^\circ \le \phi \le 270^\circ$

(a) $D_0 = \dfrac{4\pi U_{max}}{P_{rad}}, \quad U_{max} = \sin\theta\cos^2\phi\Big|_{\substack{\theta=90^\circ \\ \phi=180^\circ}} = 1$

$$P_{\text{rad}} = \int_{\pi/2}^{3\pi/2} \int_0^{\pi} U(\theta, \phi) \sin\theta \, d\theta \, d\phi = \int_{\pi/2}^{3\pi/2} \int_0^{\pi} \sin\theta \cos^2\phi \sin\theta \, d\theta \, d\phi$$

$$= \int_{\pi/2}^{3\pi/2} \cos^2\phi \, d\phi \int_0^{\pi} \sin^2\theta \, d\theta$$

$$P_{\text{rad}} = \left(\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$D_0 = \frac{4\pi(1)}{\pi^2/4} = \frac{16}{\pi} = 5.09296 = 10\log_{10}(5.09296) = 7.0697 \text{ dB}$$

$$\boxed{D_0(\text{exact}) = 5.09296(\text{dim}) = 7.0697 \text{ dB}}$$

(b) Azimuth (Horizontal) Principal Plane ($\theta = 90°$):

$$U(\theta = 90°) = \sin\theta \cos^2\phi|_{\theta=90°} = \cos^2\phi$$

$$U_h = \cos^2\phi|_{\phi=\phi_h} = 0.5 \Rightarrow \phi_h = \cos^{-1}(\pm\sqrt{0.5}) = \cos^{-1}(\pm 0.707) = 135°$$

$$\Phi_h(az) = 2(180 - 135) = 2(45°) = 90°$$

$$\boxed{\Phi_h(az) = 90°}$$

(c) Elevation (vertical) Principal plane ($\phi = 180°$):

$$U(\phi = 180°) = \sin\theta \cos^2\phi|_{\phi=180°} = \sin\theta$$

$$U_h = \sin\theta|_{\theta=\theta_h} = 0.5 \Rightarrow \theta_h = \sin^{-1}(0.5) = 30°$$

$$\Theta_h = 2(90° - 30°) = 2(60) = 120°$$

$$\boxed{\Theta_h \text{ (elev)} = 120°}$$

(d) **Either:** $D_0(\text{Kraus}) = \dfrac{41{,}253}{\Phi_h\Theta_h} = \dfrac{41{,}253}{90°(120°)} = 3.8197 = 5.82 \text{ dB}$

$$\boxed{D_0(\text{Kraus}) = 3.8197 \text{ dim} = 5.82 \text{ dB}}$$

or:

$$D_0(\text{Tai \& Pereira}) = \frac{72{,}815}{\Phi_h^2 + \Theta_h^2} = \frac{72{,}815}{(90°)^2 + (120°)^2} = \frac{72{,}815}{25{,}500} = 3.236$$

$$\boxed{D_0(\text{T\&P}) = 3.236 \text{ dim} = 5.1 \text{ dB}}$$

2.14. Using the half-power beamwidths found in Problem 2.12, the directivity for each intensity using Kraus' and Tai & Pereira's formulas is given by
$U = \sin\theta \sin\phi$;

(a) $D_0 \simeq \dfrac{41253}{\Theta_{1d}\Theta_{2d}} = \dfrac{41253}{120(120)} = 2.86 = 4.57 \text{ dB}$

(b) $D_0 \simeq \dfrac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \dfrac{72{,}815}{(120)^2 + (120)^2} = 2.53 = 4.03$ dB

$U = \sin\theta \sin^2\phi$;
(a) $D_0 \simeq 3.82 = 5.82$ dB
(b) $D_0 \simeq 3.24 = 5.10$ dB
$U = \sin\theta \sin^3\phi$;
(a) $D_0 \simeq 4.59 = 6.62$ dB
(b) $D_0 \simeq 3.64 = 5.61$ dB
$U = \sin^2\theta \sin\phi$;
(a) $D_0 \simeq 3.82 = 5.82$ dB
(b) $D_0 \simeq 3.24 = 5.10$ dB
$U = \sin^2\theta \sin^2\phi$;
(a) $D_0 \simeq 5.09 = 7.07$ dB
(b) $D_0 \simeq 4.49 = 6.53$ dB
$U = \sin^2\theta \sin^3\phi$;
(a) $D_0 \simeq 6.12 = 7.87$ dB
(b) $D_0 \simeq 5.31 = 7.25$ dB

2.15. (a) $D_0 = \dfrac{4\pi}{\Theta_{1r}\Theta_{2r}} = \dfrac{4\pi}{(1.5064)^2} = 5.5377 = 7.433$ dB

(b) $D_0 = \dfrac{32\ln(2)}{\Theta_{1r}^2 + \Theta_{2r}^2} = \dfrac{32\ln(2)}{(1.5064)^2 + (1.5064)^2} = 4.88725 = 6.8906$ dB

2.16. (a) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} = \dfrac{U_{\max}}{U_0}$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi} U \ \sin\theta \, d\theta = 2\pi \left\{ \int_0^{30^\circ} \sin\theta \, d\theta \right.$$

$$\left. + \int_{30^\circ}^{60^\circ} (0.5) \ \sin\theta \, d\theta + \int_{60^\circ}^{90^\circ} (0.1) \ \sin\theta \, d\theta \right\}$$

$$= 2\pi \left\{ (-\cos\theta)\Big|_0^{30^\circ} + \left(-\frac{\cos\theta}{2}\right)\Big|_{30^\circ}^{60^\circ} + (-0.1\cos\theta)\Big|_{60^\circ}^{90^\circ} \right\}$$

$$= 2\pi \left\{ (-0.866 + 1) + \left(\frac{-0.5 + 0.866}{2}\right) + \left(\frac{-0 + 0.5}{10}\right) \right\}$$

$$P_{\text{rad}} = 2\pi\{-0.866 + 1 - 0.25 + 0.433 + 0.05\} = 2\pi(0.367)$$

$$= 0.734\pi = 2.3059$$

$$D_0 = \frac{1(4\pi)}{2.3059} = 5.4496 = 7.3636 \text{ dB}$$

(b) $D_0(\text{dipole}) = 1.5 = 1.761$ dB
$D_0(\text{above dipole}) = (7.3636 - 1.761)$ dB $= 5.6026$ dB
$D_0(\text{above dipole}) = \frac{5.45}{1.5} = 3.633 = 5.603$ dB

2.17. (a) $P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = \int_0^{2\pi} \sin^2\phi\, d\phi \int_0^{\pi/2} \cos^4\theta\ \sin\theta\, d\theta$

$$= (\pi)\left(\frac{1}{5}\right) = \frac{\pi}{5}$$

$$U_{\max} = U(\theta = 0^\circ,\ \phi = \pi/2) = 1$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi}{(\pi/5)} = 20 = 13.0\text{ dB}$$

(b) Elevation Plane: θ varies, ϕ fixed

$\Rightarrow$ Choose $\phi = \pi/2$.

$U(\theta, \phi = \pi/2) = \cos^4\theta, \quad 0 \le \theta \le \pi/2.$

$$\cos^4\left[\frac{\text{HPBW(el.)}}{2}\right] = \frac{1}{2}$$

$$\text{HPBW(el.)} = 2\cos^{-1}\{\sqrt{0.5}\}^{1/2} = 65.5^\circ$$

2.18. (a) $P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = 2\pi$

$$\cdot\left\{\int_0^{30^\circ} \sin\theta\, d\theta + \int_{30^\circ}^{90^\circ} \frac{\cos\theta\sin\theta}{0.866} d\theta\right\}$$

$$= 2\pi\left\{\int_0^{\pi/6} \sin\theta\, d\theta + \int_{\pi/6}^{\pi/2} \frac{1}{0.866}\cos\theta\sin\theta\, d\theta\right\}$$

$$= 2\pi\left\{-\cos\theta\Big|_0^{\pi/6} + \frac{1}{0.866}\left(-\frac{\cos^2\theta}{2}\right)\Big|_{\pi/6}^{\pi/2}\right\} = 2\pi[-0.866 + 1 + 0.433]$$

$$P_{\text{rad}} = 3.5626$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(1)}{3.5626} = 3.5273 = 5.4745\text{ dB}$$

(b) $U = \dfrac{\cos(\theta)}{0.866} = 0.5 \Rightarrow \cos\theta = 0.5(0.866) = 0.433, \theta = \cos^{-1}(0.433) = 64.34^\circ$

$$\Theta_{1r} = 2(64.34) = 128.68^\circ = 2.246\text{ rad} = \Theta_{2r}$$

$$D_0 \simeq \frac{4\pi}{\Theta_{1r}\Theta_{2r}} = \frac{4\pi}{(2.246)^2} = 2.4912 = 3.9641\text{ dB}$$

2.19. (a) 35 dB

(b) $20\log_{10}\left|\dfrac{E_{\max}}{E_s}\right| = 35,\ \log_{10}\left|\dfrac{E_{\max}}{E_s}\right| = \dfrac{35}{20} = 1.75$

$$\left|\frac{E_{\max}}{E_s}\right| = 10^{1.75} = 56.234$$

2.20. (a) $U = \sin\theta, U_{\max} = 1, \quad P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^2\theta \, d\theta \, d\phi = \pi^2$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.2732$$

(b) HPBW $= 120°, 2\pi/3$
The directivity based on (2-33a) is equal to,

$$D_0 = \frac{101}{120° - 0.0027(120°)^2} = 1.2451$$

while that based on (2-33b) is equal to,

$$D_0 = -172.4 + 191\sqrt{0.818 + \frac{1}{120°}} = 1.2245$$

(c) Computer Program: $D_0 = 1.2732$

2.21. (a) $U = \sin^3\theta, U_{\max} = 1, P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} \sin^4\theta \, d\theta \, d\phi = \frac{3}{4}\pi^2$

$$D_0 = \frac{4\pi}{\frac{3}{4}\pi^2} = \frac{16}{3\pi} = 1.6976$$

(b) HPBW $= 74.93°$
From (2-33a), $D_0 = \dfrac{101}{(74.93°) - 0.0027(74.93)°)^2} = 1.68971$
From (2-33b), $D_0 = -172.4 + 191\sqrt{0.818 + \dfrac{1}{74.93°}} = 1.75029$

(c) Computer program: $D_0 = 1.693$
The value of $D_0 = 1.693$ is similar to that of (4-91) or 1.643

2.22. (a) $U = J_1{}^2(\text{ka}\sin\theta)$,
$a = \lambda/10, \text{ka}\sin\theta = \dfrac{\pi}{5}\sin\theta$. HPBW $= 93.10°$
From (2-33a): $D_0 = 101/[(93.10) - 0.0027(93.10)^2] = 1.449120$
From (2-33b): $D_0 = -172.4 + 191\sqrt{0.818 + \dfrac{1}{93.10}} = 1.477271$

$$a = \lambda/20, ka \sin\theta = \frac{\pi}{10}\sin\theta, \text{ HPBW} = 91.10°$$

From (2-33a), $D_0 = 1.47033$; From (2-33b), $D_0 = 1.502$

(b) $a = \dfrac{\lambda}{10}$: $P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} J_1^2(\text{ka}\sin\theta) \cdot \sin\theta \, d\theta \, d\phi = 0.7638045$

$$U_{\max} = 0.0893, \; D_0 = \frac{4\pi(0.0893)}{0.7638045} = 1.469193$$

$$a = \frac{\lambda}{20}: \; P_{rad} = \int_0^{2\pi} \int_0^{\pi} J_1{}^2(\pi/10 \sin\theta) \sin\theta \, d\theta \, d\phi = 0.202604$$

$$U_{max} = 0.0240714, \; D_0 = \frac{4\pi(0.0240714)}{0.202604} = 1.49257$$

If the radius of loop is smaller than $\lambda/20$, the directivity approaches 1.5.

2.23. Using the numerical techniques, the directivity for each intensity of Prob. 2.12, with 10° uniform divisions is equal to for $U = \sin\theta \sin\phi$:

(a) Midpoint: $D_0 = \dfrac{4\pi U_{max}}{P_{rad}}$

$$U_{max} = 1: P_{rad} = \frac{\pi}{18}\left(\frac{\pi}{18}\right) \sum_{j=1}^{18} \sin\phi_j \sum_{i=1}^{18} \sin^2\theta_i$$

$$\theta_i = \frac{\pi}{36} + (i-1)\frac{\pi}{18}, \; i = 1, 2, 3, ..., 18$$

$$\phi_j = \frac{\pi}{36} + (j-1)\frac{\pi}{18}, \; j = 1, 2, 3, ..., 18$$

$$P_{rad} = \left(\frac{\pi}{18}\right)^2 (11.38656)(8.9924) = 3.119$$

$$D_0 = \frac{4\pi(1)}{3.119} = 4.03 = 6.05 \text{ dB}$$

(b) Trailing edge of each division:

$$\text{Trailing edge: } \theta_i = i(\pi/18), i = 1, 2, 3, \ldots, 18$$

$$\phi_j = j(\pi/18), j = 1, 2, 3, \ldots, 18$$

$$P_{rad} = \left(\frac{\pi}{18}\right)^2 (11.25640)(8.96985) = 3.076$$

$$D_0 = \frac{4\pi(1)}{3.119} = 4.09 = 6.11 \text{ dB}$$

In a similar manner:

$U = \sin\theta \sin^2\phi$;
(a) $P_{rad} = 2.463 \Rightarrow D_0 = 5.10 = 7.07$ dB
(b) $P_{rad} = 2.451 \Rightarrow D_0 = 5.13 = 7.10$ dB

$U = \sin\theta \sin^3\phi$;
(a) $P_{rad} = 2.092 \Rightarrow D_0 = 6.01 = 7.79$ dB
(b) $P_{rad} = 2.086 \Rightarrow D_0 = 6.02 = 7.80$ dB

$U = \sin^2\theta \sin\phi$;
(a) $P_{rad} = 2.469 \Rightarrow D_0 = 4.74 = 6.76$ dB
(b) $P_{rad} = 2.618 \Rightarrow D_0 = 4.80 = 6.81$ dB

$U = \sin^2\theta \sin^2\phi$;
(a) $P_{rad} = 2.092 \Rightarrow D_0 = 6.01 = 7.79$ dB
(b) $P_{rad} = 2.086 \Rightarrow D_0 = 6.02 = 7.80$ dB

$U = \sin^2\theta \sin^3\phi$;
(a) $P_{rad} = 1.777 \Rightarrow D_0 = 7.07 = 8.49$ dB
(b) $P_{rad} = 1.775 \Rightarrow D_0 = 7.08 = 8.50$ dB

2.24. Using the computer program **Directivity** of Chapter 2, the directivities for each radiation intensity of Problem 2.12 are equal to:

(a) $U = \sin\theta \sin\phi$; $P_{rad} = 3.1318$

$U_{max} = 1; \quad D_0 = \dfrac{4\pi \cdot U_{max}}{3.1318} = 4.0125 \Rightarrow 6.034 \text{ dB}$

(b) $U = \sin\theta \sin^2\phi$; $P_{rad} = 2.4590$

$U_{max} = 1; \quad D_0 = \dfrac{4\pi \cdot 1}{2.4590} = 5.110358 \Rightarrow 7.0845 \text{ dB}$

(c) $U = \sin\theta \sin^3\phi$; $P_{rad} = 2.0870$

$U_{max} = 1; \quad D_0 = \dfrac{4\pi \cdot 1}{2.0870} = 6.02124 \Rightarrow 7.80 \text{ dB}$

(d) $U = \sin^2\theta \sin\phi$; $P_{rad} = 2.6579$

$U_{max} = 1; \quad D_0 = \dfrac{4\pi \cdot 1}{2.6579} = 4.72793 \Rightarrow 6.746 \text{ dB}$

(e) $U = \sin^2\theta \sin^2\phi$; $P_{rad} = 2.0870$

$U_{max} = 1; \quad D_0 = \dfrac{4\pi \cdot 1}{2.0870} = 6.02126 \Rightarrow 7.7968 \text{ dB}$

(f) $U = \sin^2\theta \sin^3\phi$; $P_{rad} = 1.7714$

$U_{max} = 1; \quad D_0 = \dfrac{4\pi \cdot 1}{1.7714} = 7.09403 \Rightarrow 8.5089 \text{ dB}$

2.25. (a) $E|_{max} = \cos\left[\dfrac{\pi}{4}(\cos\theta - 1)\right]|_{max} = 1$ at $\theta = 0°$.

$$0.707 E_{max} = 0.707 \cdot (1) = \cos\left[\frac{\pi}{4}(\cos\theta_1 - 1)\right]$$

$$\frac{\pi}{4}(\cos\theta_1 - 1) = \pm\frac{\pi}{4} \Rightarrow \theta_1 = \begin{cases} \cos^{-1}(2) = \text{does not exist} \\ \cos^{-1}(0) = 90° = \dfrac{\pi}{2} \text{ rad.} \end{cases}$$

$$\Theta_{1r} = \Theta_{2r} = 2\left(\frac{\pi}{2}\right) = \pi$$

$$D_0 \simeq \frac{4\pi}{\Theta_{1r}\Theta_{2r}} = \frac{4\pi}{\pi^2} = \frac{4}{\pi} = 1.273 = 1.049 \text{ dB}$$

(b) Using the computer program **Directivity** of Chapter 2

$$D_0 = 2.00789 = 3.027 \text{ dB}$$

Since the pattern is not very narrow, the answer obtained using Kraus' approximate formula is not as accurate.

2.26. (a) $E\Big|_{max} = \cos\left[\dfrac{\pi}{4}(\cos\theta + 1)\right]\Big|_{max} = 1$ at $\theta = \pi$.

$0.707 = \cos\left[\dfrac{\pi}{4}(\cos\theta_1 + 1)\right]$

$\dfrac{\pi}{4}(\cos\theta_1 + 1) = \pm\dfrac{\pi}{4} \Rightarrow \theta_1 = \begin{cases} \cos^{-1}(-2) \rightarrow \text{does not exist.} \\ \cos^{-1}(0) \rightarrow 90° \rightarrow \dfrac{\pi}{2} \text{ rad} \end{cases}$

$\Theta_{1r} = \Theta_{2r} = 2\left(\dfrac{\pi}{2}\right) = \pi$

$D_0 \simeq \dfrac{4\pi}{\pi^2} = \dfrac{4}{\pi} = 1.273 = 1.049 \text{ dB}$

(b) Computer Program **Directivity**:
$D_0 = 2.00789 = 3.027$ dB

2.27. (a) $P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi/2} U_0 \sin(\pi \sin\theta)\sin\theta \, d\theta \, d\phi = 2\pi U_0 \frac{\pi}{2} J_1(\pi) = U_0 \pi^2 J_1(\pi)$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi U_0}{U_0 \pi^2 J_1(\pi)} = \frac{4}{\pi}\frac{1}{J_1(\pi)} = 4.4735$$

$$\frac{\pi}{2} J_1(\pi) = 0.447$$

(b) Computer program **Directivity**:

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi/2} U_0 \sin(\pi \sin\theta)\sin\theta \, d\theta \, d\phi = 2\pi(0.447)$$

$$D_0 = 4.4735$$

2.28. $E_\phi = C_0 \sin^{1.5}\theta \frac{e^{-jkr}}{r}$

(a) $U_n = |E_\phi|^2 = e_0^2 \sin^3\theta, \Rightarrow U_{n|\max} = C_0^2$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi = 2\pi \int_0^{\pi} C_0^2 \sin^3\theta \ \sin\theta \, d\theta = C_0(2\pi)\int_0^{\pi} \sin^4\theta \, d\theta$$

$$\int_0^{\pi} \sin^4\theta \, d\theta = -\frac{\sin^3\theta\cos\theta}{4}\Big|_0^{\pi} + \frac{4-3}{4}\int_0^{\pi} \sin^2\theta \, d\theta = \frac{3}{4}\int_0^{\pi} \sin^2\theta \, d\theta$$

$$= \frac{3}{4}\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right]_0^{\pi} = \frac{3}{4}\left(\frac{\pi}{2}\right) = \frac{3\pi}{8}$$

$$P_{\text{rad}} = 2\pi C_0^2 \left(\frac{3\pi}{8}\right) = \frac{3\pi^2}{4} C_0^2$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi C_0^2}{\frac{3\pi^2}{4} C_0^2} = \frac{16}{3\pi} = 1.69765 = 2.298 \text{ dB}$$

$$\boxed{D_0 = 1.69765 = 2.298 \text{ dB}}$$

(b) $U_n = C_0 \sin^3\theta, \ U_n\Big|_{\max} = C_0^2 \ \text{ at } \theta = 90°, \ U_n\Big|_{\theta=\theta_n} = 0.5C_0^2 = \sin^3\theta_h C_0^2$

$$\sin^3\theta_h = 0.5, \theta_h = \sin^{-1}(0.5)^{1/3} = \sin^{-1}(0.7937) = 52.5327°$$

$$\Theta_h = 2(90° - 52.5327°) = 74.935°$$

$$D_0(\text{McDonald}) = \frac{101}{74.935 - 0.0027(74.935)^2} = \frac{101}{59.7738} = \boxed{1.6897 = 2.278 \text{ dB}}$$

$$D_0(\text{McDonald}) = \boxed{1.6897 \text{ dimensionless} = 2.278 \text{ dB}}$$

$$D_0(\text{Pozar}) = -172.4 + 191\sqrt{0.818 + \frac{1}{74.935°}} = -172.4 + 191(0.91178)$$

$$D_0(\text{Pozar}) = -172.4 + 174.1502 = \boxed{1.7502 \text{ dimensionless} = 2.431 \text{ dB}}$$

2.29. (a) Using the computer program **Directivity** of Chapter 2.

$$D_0 = 14.0707 \text{ dimensionless} = 11.48 \text{ dB}$$

(b) $U\,|_{max} = \left[\dfrac{\sin(\pi \sin\theta)}{\pi\sin\theta}\right]^2_{max} = 1 \quad \text{when } \theta = 0°.$

$$U = \frac{1}{2}U_{max} = \frac{1}{2}(1) = \left[\frac{\sin(\pi\sin\theta_1)}{\pi\sin\theta_1}\right]^2$$

Iteratively we obtain $\theta_1 = 26.3°$. Therefore

$$\Theta_{1d} = \Theta_{2d} = 2(26.3°) = 52.6°.$$

and $D_0 \simeq \dfrac{41,253}{(52.6)^2} = 14.91$ dimensionless $= 11.73$ dB using the Kraus' formula

(c) For Tai and Pereira's formula

$$D_0 = \frac{72{,}815}{2\cdot\Theta_{1d}^2} = \frac{72{,}815}{2(52.6)^2} = 13.16 \text{ dimensionless} = 11.19 \text{ dB}$$

2.30. $U = \dfrac{1}{2\eta}|E|^2 = \dfrac{1}{2\eta}\sin\theta\cos^2\phi \Rightarrow U_{max} = \dfrac{1}{2\eta}$

(a) $P_{rad} = 2\cdot\displaystyle\int_0^{\pi/2}\int_0^{\pi}\frac{1}{2\eta}\sin^2\theta\cos^2\phi\, d\theta\, d\phi = \frac{1}{\eta}\left(\frac{\pi}{4}\right)\left(\frac{\pi}{2}\right) = \frac{\pi^2}{8\eta}$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi\left(\dfrac{1}{2\eta}\right)}{\dfrac{\pi^2}{8\eta}} = \frac{16}{\pi} = 5.09 = 7.07 \text{ dB}$$

(b) $U_{max} = \dfrac{1}{2\eta}$ at $\theta = \pi/2, \phi = 0$

In the elevation plane through the maximum $\phi = 0$ and $U = \dfrac{1}{2\eta}\sin\theta$, the 3-dB point occurs when

$$U = 0.5U_{max} = 0.5\left(\frac{1}{2\eta}\right) = \frac{1}{2\eta}\sin\theta_1 \Rightarrow \theta_1 = \sin^{-1}(0.5) = 30°$$

Therefore $\Theta_{1d} = 2(90 - 30) = 120°$

In the azimuth plane through the maximum $\theta = \pi/2$ and $U = \dfrac{1}{2\eta}\cos^2\phi$, the 3-dB point occurs when $U = 0.5U_{max} = 0.5\left(\dfrac{1}{2\eta}\right) = \dfrac{1}{2\eta}\cos^2\theta_1 \Rightarrow \phi_1 = \cos^{-1}(0.707) = 45°$

$$\Theta_{2d} = 2(90° - 45°) = 90°$$

Therefore using Kraus' formula: $D_0 \simeq \dfrac{41{,}253}{120(90)} = 3.82$ dimensionless $= 5.82$ dB

(c) Using Tai and Pereira's formula:

$$D_0 \simeq \frac{72{,}815}{\Theta_{1d}^2 + \Theta_{2d}^2} = \frac{72{,}815}{(120)^2 + (90)^2} = 3.24 \text{ dimensionless} = 5.10 \text{ dB}$$

(d) Using the computer program **Directivity** of Chapter 2.

$$D_0 = 5.16425 = 7.13 \text{ dB}$$

2.31. $U = \left[\frac{J_1(ka\sin\theta)}{\sin\theta}\right]^2 = (ka)^2\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 = U_0\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2$

(a) $U_{\max} = U_0\left(\frac{1}{2}\right)^2 = \frac{U_0}{4}$ and it occurs when $ka\sin\theta = 0 \Rightarrow \theta = 0°$. The 3-dB point is obtained using

$$U = \frac{1}{2}U_{\max} = \frac{U_0}{8} = U_0\left[\frac{J_1(ka\sin\theta)}{ka\sin\theta}\right]^2 \Rightarrow \frac{J_1(ka\sin\theta)}{ka\sin\theta} = 0.3535$$

with the aid of the $J_1(x)/x$ of Appendix V.

$$x = ka\sin\theta_1 = 1.61 \Rightarrow \theta_1 = \sin^{-1}(1.61/2\pi) = 14.847°$$
$$\Rightarrow \Theta_{1r} = 29.694°$$

(b) Since $\Theta_{1r} = \Theta_{2r} = 29.694°$, the directivity using Kraus' formula is equal to

$$D_0 \simeq \frac{41,253}{(29.694)^2} = 46.79 \text{ dimensionless} = 16.70 \text{ dB}$$

2.32. $G_0 = 16 \text{ dB} \Rightarrow 16 = 10\log_{10} G_0 \text{ (dimensionless)} \Rightarrow G_0\text{(dimensionless)} = 10^{1.6} = 39.81$

$r = 100 \text{ meters} = 10,000 \text{ cm} = 10^4 \text{ cm}$

$P_{\text{rad}} = e_{cd}P_{in} = (1)P_{in} = 8 \text{ watts}$

$f = 1,900 \text{ MHz} \Rightarrow \lambda = 30\times 10^9/1.9\times 10^9 = 15.789 \text{ cm}$

(a)
$$W_0 = \frac{P_{\text{rad}}}{4\pi r^2} = \frac{8}{4\pi(10^4)^2} = \frac{8}{4\pi\times 10^8}$$
$$= \frac{2}{\pi}\times 10^{-8} = 0.6366\times 10^{-8} \text{ Watts/cm}^2$$
$$W_0 = 0.6366\times 10^{-8} = 6.366\times 10^{-9} \text{ Watts/cm}^2$$
$$W_{\max} = W_0 G_0(\text{dim}) = 6.366\times 10^{-9}(39.81) = 253.438\times 10^{-9}.$$

$$\boxed{W_{\max} = 253.438\times 10^{-9} \text{ Watts/cm}^2}$$

(b) $D_0(\lambda/4 \text{ monopole}) = 1.643$

$$A_{em} = \frac{\lambda^2}{4\pi}D_0 = \frac{\lambda^2}{4\pi}(1.643) = \frac{1.643(15.789)^2}{4\pi} = 32.5938 \text{ cm}^2$$
$$A_{em} = 32.5938 \text{ cm}^2$$
$$P(\text{received}) = W_{\max}A_{em} = (253.438\times 10^{-9})(32.5938)$$

$$\boxed{P(\text{received}) = 8.2606\times 10^{-6} \text{ Watts}}$$

2.33. (a) Linear because $\Delta\phi = 0$.
(b) Linear because $\Delta\phi = 0$.

(c) Circular because
1. $E_x = E_y$
2. $\Delta\phi = \pi/2$.
CCW because E_y leads E_x. AR = 1, $\tau = 90°$

(d) Circular because
1. $E_x = E_y$
2. $\Delta\phi = -\pi/2$
CW because E_y lags E_x. AR = 1, $\tau = 90°$

(e) Elliptical because $\Delta\phi$ is not odd multiples of $\pi/2$. CCW because E_y leads E_x.
AR = OA/OB
Letting $E_x = E_y = E_0$

$$\left.\begin{aligned} \text{OA} &= E_0[0.5(1+1+\sqrt{2})]^{1/2} = 1.30656E_0 \\ \text{OB} &= E_0[0.5(1+1+\sqrt{2})]^{1/2} = 0.541196E_0 \end{aligned}\right\} \Rightarrow \text{AR} = \frac{1.30656}{0.541196} = 2.414$$

$$\tau = 90° - \frac{1}{2}\tan^{-1}\left[\frac{2(1)\cos(45°)}{1-1}\right] = 90° - \frac{1}{2}\tan^{-1}\left(\frac{1.414}{0}\right)$$

$$= 90° - \frac{1}{2}(90°) = 45°$$

(f) Elliptical because $\Delta\phi$ is not odd multiples of $\pi/2$. CW because E_y lags E_x.

$$\text{From above}\ \left.\begin{aligned} \text{OA} &= 1.30656E_0 \\ \text{OB} &= 0.541196E_0 \end{aligned}\right\} \Rightarrow \text{AR} = \frac{1.30656}{0.541196} = 2.414$$

From above $\tau = 90° - \frac{1}{2}(90°) = 45°$

(g) Elliptical because
1. $E_x \neq E_y$
2. $\Delta\phi$ is not zero or multiples of π.
CCW because E_y leads E_x.

$$\left.\begin{aligned} \text{OA} &= E_y\{\tfrac{1}{2}[0.25+1+0.75]\}^{1/2} = E_y \\ \text{OB} &= E_y\{\tfrac{1}{2}[0.25+1-0.75]\}^{1/2} = 0.5E_y \end{aligned}\right\} \Rightarrow \text{AR} = \frac{1}{0.5} = 2$$

$$\tau = 90° - \frac{1}{2}\tan^{-1}\left(\frac{0}{-0.75}\right) = 90° - \frac{1}{2}(180°) = 0°$$

(h) Elliptical because
1. $E_x \neq E_y$
2. $\Delta\phi$ is not zero or multiples of π.
CCW because E_y lags E_x.

$$\text{From above}\ \left.\begin{aligned} \text{OA} &= E_y \\ \text{OB} &= 0.5E_y \end{aligned}\right\} \Rightarrow \text{AR} = \frac{1}{0.5} = 2$$

$$\tau = 90° - \frac{1}{2}(180°) = 0°$$

2.34. $\mathcal{E}_x(z,t) = \text{Re}[E_x e^{j(\omega t + kz + \phi_x)}] = E_x\cos(\omega t + kz + \phi_x)$

$\mathcal{E}_y(z,t) = \text{Re}[E_y e^{j(\omega t + kz + \phi_y)}] = E_y\cos(\omega t + kz + \phi_y)$

where E_x and E_y are real positive constants.

Choosing $z = 0$ and letting $\Delta\phi = \phi_y - \phi_x = \phi_y - 0 = \phi$

$$\mathcal{E}_x(t) = E_x \cos(\omega t) \tag{1}$$
$$\mathcal{E}_y(t) = E_y \cos(\omega t + \phi)$$

and

$$\mathcal{E}(t) = \sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2} = \sqrt{E_x^2 \cos^2(\omega t) + E_y^2 \cos^2(\omega t + \phi)} \tag{2}$$

The maximum and minimum values of (2) are the major and minor axes of the polarization ellipse. Squaring (2) and using the half-angle identity, (2) can be written as

$$\mathcal{E}^2(t) = \frac{1}{2}\{E_x^2 + E_y^2 + E_x^2 \cos(2\omega t) + E_y^2 \cos^2[2(\omega t + \phi)]\} \tag{3}$$

Since E_x and E_y are constants, the maximum and minimum values of (3) occur when $f(t) = {E_x}^2 \cos(2\omega t) + {E_y}^2 \cos[2(\omega t + \phi)]$ is maximum or minimum. These are found by differentiating (4) and setting it equal to zero. Thus

$$\frac{df}{d(2\omega t)} = -{E_x}^2 \sin(2\omega t) - E_y^2 \sin[2(\omega t + \phi)] = 0 \tag{4}$$

or

$$E_x^2 \sin(2\omega t) = -E_y^2 \sin[2(\omega t + \phi)]$$
$$= -E_y^2\{\sin 2\omega t \cos 2\phi + \cos 2\omega t \sin 2\phi\} \tag{5}$$

Dividing (5) by $\cos(2\omega t)$ yields

$$E_x^2 \tan(2\omega t) = -E_y^2[\tan(2\omega t)\cos(2\phi) + \sin(2\phi)]$$

or

$$\tan(2\omega t) = \frac{-E_y^2 \sin(2\phi)}{E_x^2 + E_y^2 \cos(2\phi)}$$

from which we obtain that

$$\cos(2\omega t) = \frac{E_x^2 + E_y^2 \cos(2\phi)}{\pm\rho} \tag{6}$$
$$\cos(2\omega t + 2\phi) = \frac{E_y^2 + E_x^2 \cos(2\phi)}{\pm\rho} \tag{7}$$

where

$$\rho = \sqrt{E_x^4 + E_y^4 + 2E_x^2 E_y^2 \cos(2\phi)} \tag{8}$$

Substituting (6)–(8) into (3) yields

$$\mathcal{E}^2 = \frac{1}{2}\left[E_x^2 + E_y^2 \pm \frac{1}{\rho}(\rho^2)\right]$$

whose maximum value is

$$\mathcal{E}_{\text{max}} = \text{OA} = \left\{\frac{1}{2}[E_x^2 + E_y^2 + (E_x^4 + E_y^4 + 2E_x^2E_y^2\cos 2\phi)^{1/2}]\right\}^{1/2}$$

$$\mathcal{E}_{\text{max}} = \text{OB} = \left\{\frac{1}{2}[E_x^2 + E_y^2 - (E_x^4 + E_y^4 + 2E_x^2E_y^2\cos 2\phi)^{1/2}]\right\}^{1/2}$$

The tilt angle τ can be obtained by expanding (1) and writing the two as

$$\frac{\mathcal{E}_x^2}{E_x^2} - \frac{2\mathcal{E}_x\mathcal{E}_y\cos\phi}{E_xE_y} + \frac{\mathcal{E}_y^2}{E_y^2} = \sin^2\phi \tag{9}$$

which is the equation of a tilted ellipse. Choosing a coordinate system whose principal axes coincide with the major and minor axes of the tilted ellipse, we can write that

$$\begin{aligned}\mathcal{E}_x &= \mathcal{E}_x'\sin(z) - \mathcal{E}_y'\cos(z)\\ \mathcal{E}_y &= \mathcal{E}_x'\cos(z) + \mathcal{E}_y'\sin(z)\end{aligned} \tag{10}$$

where $\mathcal{E}_x'$ and $\mathcal{E}_y'$ are the new field values along the new principal axes x', y', z'. Substituting (10) into (9) yields

$$\frac{2\mathcal{E}_x'\mathcal{E}_y'\cos(z)\sin(z)}{E_x^2} - \frac{2\mathcal{E}_x'\mathcal{E}_y'\cos(z)\sin(z)}{E_y^2} - \frac{2\mathcal{E}_x'\mathcal{E}_y'\cos\phi}{E_xE_y}(\sin^2 z - \cos^2 z) = 0$$

which when solved for the tilt angle τ reduces to

$$\tan\left[2\left(\frac{\pi}{2} - \tau\right)\right] = \frac{2E_xE_y\cos\phi}{E_x^2 - E_y^2}$$

or

$$\tau = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{2E_xE_y\cos\phi}{E_x^2 - E_y^2}\right)$$

For more details on the tilt angle derivation, see J.D. Kraus, *Antennas*, McGraw-Hill, 1950, pp. 464–476.

2.35. (a) $\hat{\rho}_w = \hat{a}_x\cos\phi_1 + \hat{a}_y\sin\phi_1$

$$\hat{\rho}_a = \hat{a}_x\cos\phi_2 + \hat{a}_y\sin\phi_2$$

$$\begin{aligned}\text{PLF} &= |\hat{\rho}_w\cdot\hat{\rho}_a|^2 = |(\hat{a}_x\cos\phi_1 + \hat{a}_y\sin\phi_1)\cdot(\hat{a}_x\cos\phi_2 + \hat{a}_y\sin\phi_2)|^2\\ &= |\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2|^2 = |\cos(\phi_1 - \phi_2)|^2\end{aligned}$$

(b) $\hat{\rho}_w = \hat{a}_x \sin\theta_1 \cos\phi_1 + \hat{a}_y \sin\theta_1 \sin\phi_1 + \hat{a}_z \cos\theta_1$

$\hat{\rho}_a = \hat{a}_x \sin\theta_2 \cos\phi_2 + \hat{a}_y \sin\theta_2 \sin\phi_2 + \hat{a}_z \cos\theta_2$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |\sin\theta_1 \cos\phi_1 \sin\theta_2 \cos\phi_2 + \sin\theta_1 \sin\phi_1 \sin\theta_2 \cdot \sin\phi_2 + \cos\theta_1 \cdot \cos\theta_2|^2$$

$$\text{PLF} = |\sin\theta_1 \cdot \sin\theta_2(\cos\phi_1 \cdot \cos\phi_2 + \sin\phi_1 \sin\phi_2) + \cos\theta_1 \cos\theta_2|^2$$

$$\text{PLF} = |\sin\theta_1 \sin\theta_2 \cos(\phi_1 - \phi_2) + \cos\theta_1 \cos\theta_2|^2$$

2.36. Assuming electric field is x-polarized

(a) $\underline{E}_w = \hat{a}_x E_1 e^{-jkz} \Rightarrow \hat{\rho}_w = \hat{a}_x$

$$\underline{E}_a = (\hat{a}_\theta - j\hat{a}_\phi)E_0 f(r,\theta,\phi) \Rightarrow \hat{\rho}_a = \left(\frac{\hat{a}_\theta - j\hat{a}_\phi}{\sqrt{2}}\right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \frac{1}{2}|\hat{a}_x \cdot \hat{a}_\theta - j\hat{a}_x \cdot \hat{a}_\phi|^2$$

since $\hat{a}_\theta = \hat{a}_x \cos\theta \cos\phi + \hat{a}_y \cos\theta \sin\phi - \hat{a}_z \sin\theta$

$\hat{a}_\phi = -\hat{a}_x \sin\phi + \hat{a}_y \cos\phi$

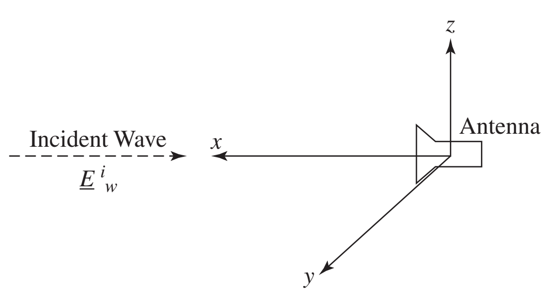
$$\text{PLF} = \tfrac{1}{2}(\cos^2\theta \cos^2\phi + \sin^2\phi)$$

(b) when $\underline{E}_a = (\hat{a}_\theta + j\hat{a}_\theta)E_0 f(r,\theta,\phi)$, PLF is also

$$\text{PLF} = \tfrac{1}{2}(\cos^2\theta \cos^2\phi + \sin^2\phi)$$

A more general, but also more complex, expression can be derived when the incident electric field is of the form $\underline{E}_w = (a\hat{a}_x + b\hat{a}_y)e^{-jkz}$ where a, b are real constants. It can be shown (using the same procedure) that

$$\text{PLF} = \frac{1}{\sqrt{2(a^2+b^2)}}[(a\cos\theta\cos\phi + b\sin\theta\sin\phi)^2 + (a\sin\phi - b\cos\phi)^2]^{1/2}$$



2.37. (a) $\underline{E}_w = E_0(j\hat{a}_y + 3\hat{a}_z)e^{+jkx}$

1. **Elliptical polarization; AR = $\frac{3}{1}$ = 3; Left Hand (CCW)**
   a. 2 components orthogonal to direction of propagation
   b. Not of same magnitude
   c. 90° phase difference between them
   d. y component is leading the z component *or* z component is lagging the y component

(b) $\underline{E}_a = E_a(\hat{a}_y + 2\hat{a}_z)e^{-jkx}$

1. **Liner polarization; AR = ∞; No rotation**
   a. 2 components orthogonal to direction of propagation.
   b. Not of same magnitude
   c. 0° phase difference between them,

(c) PLF $= |\hat{\rho}_w \cdot \hat{\rho}_a|^2$

$$\underline{E}_w = E_0(j\hat{a}_y + 3\hat{a}_z)e^{+jkx} = E_0 \underbrace{\left(\frac{j\hat{a}_y + 3\hat{a}_z}{\sqrt{10}}\right)}_{\hat{\rho}_w} \sqrt{10}e^{+jkx}$$

$$\hat{\rho}_w = \left(\frac{j\hat{a}_y + 3\hat{a}_z}{\sqrt{10}}\right)$$

$$\underline{E}_a = E_a(\hat{a}_y + 2\hat{a}_z)e^{-jkx} = E_0 \underbrace{\left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right)}_{\hat{\rho}_a} \sqrt{5}e^{-jkx}$$

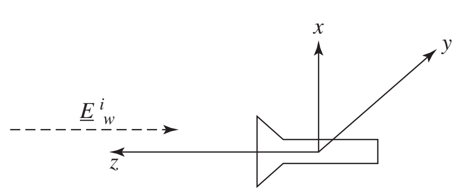
$$\hat{\rho}_a = \left(\frac{\hat{a}_y + 2\hat{a}_z}{\sqrt{5}}\right)$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{(j\hat{a}_y + 3\hat{a}_z)}{\sqrt{10}} \cdot \frac{(\hat{a}_y + 2\hat{a}_z)}{\sqrt{5}}\right|^2 = \frac{|j+6|^2}{50} = \frac{37}{50}$$

$$\text{PLF} = \frac{37}{50} = \boxed{0.740 = -1.31 \text{ dB}}$$

2.38. $\underline{E}^i_w = (\hat{a}_x + j\hat{a}_y)E_0 e^{+jkz}$

$$\underline{E}_a = (\hat{a}_x + 2\hat{a}_y)E_1 \frac{e^{-jkr}}{r}\bigg|_{\substack{\theta = 0° \\ z \text{ axis}}} = (\hat{a}_x + 2\hat{a}_y)E_1 \frac{e^{-jkz}}{z}$$



(a) $\underline{E}^i_w = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_0 e^{+jkz}$
Circular: 2 components, same amplitude, 90° phase difference

(b) **Clockwise** (y component is leading the x component)

(c) $\underline{E}_a = \left(\frac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)\sqrt{5}E_1 \frac{e^{-jkz}}{z}$
Linear: 2 components, 0° phase difference

(d) No rotation

(e) $\hat{\rho}_w = \left(\dfrac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right), \quad \hat{\rho}_a = \left(\dfrac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left[\left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right) \cdot \left(\frac{\hat{a}_x + 2\hat{a}_y}{\sqrt{5}}\right)\right]^2 = \frac{|1 + j2|^2}{10} = \frac{5}{10}$$

$$\text{PLF} = \frac{5}{10} = 0.5 = 10\log_{10}(0.5) = -3 \text{ dB}$$

2.39. (a) $\underline{E}_w = (4\hat{a}_z + j2\hat{a}_x)E_w \dfrac{e^{+jky}}{y} = \underbrace{\left(\dfrac{4\hat{a}_z + j2\hat{a}_x}{\sqrt{20}}\right)}_{\hat{\rho}_w} \sqrt{20}E_w \dfrac{e^{+jky}}{y}$

• **Elliptical** (2 components, not of same magnitude, 90° phase difference)

(b) **CW**; x-components leads z-component by 90°; rotate x into z while looking (observing) in the -y direction (from behind the wave).

(c) **AR** $= \dfrac{4}{2} = 2$

(d) $\hat{\rho}_w = \left(\dfrac{4\hat{a}_z + j2\hat{a}_x}{\sqrt{20}}\right); \ \hat{\rho}_a = \hat{a}_z$

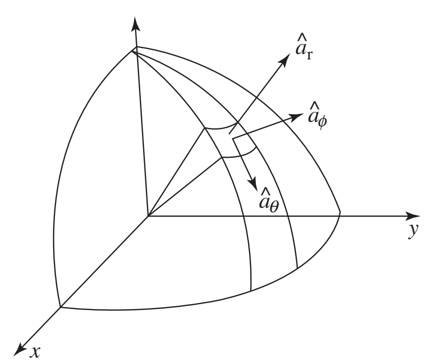
$$\textbf{PLF} = \left|\hat{\rho}_w \cdot \hat{\rho}_a\right|^2 = \left|\left(\frac{4\hat{a}_z + j2\hat{a}_x}{\sqrt{20}}\right) \cdot \hat{a}_z\right|^2 = \frac{16}{20} = 0.8 = 10\log_{10}(0.8)$$

$$\text{PLF} = 0.8 = -0.969 \text{ dB}$$

2.40. (a) $\underline{E}_a = E_0(j\hat{a}_\theta + 2\hat{a}_\phi)f_0(\theta_0, \phi_0)\dfrac{e^{-jkr}}{r} = E_0 \underbrace{\left(\dfrac{j\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{5}}\right)}_{\hat{\rho}_a} \sqrt{5}f_0(\theta_0, \phi_0)\dfrac{e^{-jkr}}{r}$

$$\hat{\rho}_a = \left(\frac{j\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{5}}\right)$$

Elliptical, CW



(b) $$\underline{E}_w = E_1(2\hat{a}_\theta + j\hat{a}_\phi)f_1(\theta_0,\phi_0)\frac{e^{+jkr}}{r}$$

$$= E_1 \underbrace{\left(\frac{2\hat{a}_\theta + j\hat{a}_\phi}{\sqrt{5}}\right)}_{\hat{\rho}_w} \sqrt{5} f_1(\theta_0,\phi_0)\frac{e^{+jkr}}{r}$$

$$\hat{\rho}_w = \left(\frac{2\hat{a}_\theta + j\hat{a}_\phi}{\sqrt{5}}\right)$$

Elliptical, CW

(c) $$\text{PLF} = |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \left|\left(\frac{j\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{5}}\right)\cdot\left(\frac{2\hat{a}_\theta + j\hat{a}_\phi}{\sqrt{5}}\right)\right|^2 = \left|\frac{2j + j2}{\sqrt{25}}\right|^2 = \left|\frac{4j}{\sqrt{25}}\right|^2$$

$$\boxed{\text{PLF} = \frac{16}{25} = 0.64 = -1.938 \text{ dB}}$$

2.41. (a) $$\underline{E}_w = E_0(\hat{a}_x \pm j\hat{a}_y)e^{-jkz} \Rightarrow \hat{\rho}_w = \frac{1}{\sqrt{2}}(\hat{a}_x \pm j\hat{a}_y)$$

$$\underline{E}_a \simeq E_1(\hat{a}_\theta - j\hat{a}_\phi)f(r,\theta,\phi) \Rightarrow \hat{\rho}_a = \frac{1}{\sqrt{2}}(\hat{a}_\theta - j\hat{a}_\phi)$$

$$\text{PLF} = \frac{1}{2}\left|(\hat{a}_x \pm j\hat{a}_y)\cdot(\hat{a}_\theta - j\hat{a}_\phi)\right|^2 = \frac{1}{2}\left|(\hat{a}_x\cdot\hat{a}_\theta \pm \hat{a}_y\cdot\hat{a}_\phi) - j(\hat{a}_x\hat{a}_\phi \mp \hat{a}_y\hat{a}_\theta)\right|^2$$

Converting the spherical unit vectors to rectangular, as it was done in Problem 2.35, leads to

$$\text{PLF} = \frac{1}{2}(\cos\theta \pm 1)^2$$

(b) When

$$\underline{E}_w = E_0(\hat{a}_x \pm j\hat{a}_y)e^{-jkz}$$
$$\underline{E}_a \simeq E_1(\hat{a}_\theta + j\hat{a}_\phi)f(r,\theta,\phi)$$

the PLF is equal to

$$\text{PLF} = \frac{1}{2}(\cos\theta \mp 1)^2$$

2.42. $\underline{E}_w = (\hat{a}_\theta \cos\phi - \hat{a}_\phi \sin\phi\cos\theta)f(r,\theta,\phi)$ or

$$\underline{E}_w = \left[\frac{\hat{a}_\theta \cos\phi - \hat{a}_\phi \sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}\right]\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}\cdot f(r,\theta,\phi)$$

Thus $$\hat{\rho}_w = \frac{\hat{a}_\theta \cos\phi - \hat{a}_\phi \sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}$$

and

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_\theta \cos\phi - \hat{a}_\phi \sin\phi\cos\theta}{\sqrt{\cos^2\phi + \sin^2\phi\cos^2\theta}}\right)\cdot \hat{a}_x\right|^2$$

Transforming the rectangular unit vector to spherical using

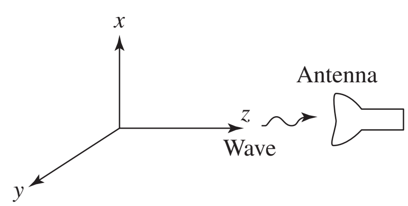
$$\hat{a}_x = \hat{a}_r \sin\theta\cos\phi + \hat{a}_\theta \cos\theta\cos\phi - \hat{a}_\phi \sin\phi$$

the PLF reduces to

$$\text{PLF} = \frac{\cos^2\theta}{\cos^2\phi + \sin^2\phi\cos^2\theta}$$

The same answer is obtained by transforming the spherical unit vectors to rectangular, as was done in Prob. 2.35.

2.43. $\underline{E}_a \simeq (2\hat{a}_x \pm j\hat{a}_y) f(r,\theta,\phi) = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}\right)\sqrt{5} f(r,\theta,\phi)$



(a) $\hat{\rho}_w = \left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right) \Rightarrow$ Wave is Right Hand (RH)

$\hat{\rho}_a = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}\right)$

$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2$

$$= \begin{cases} \frac{9}{10} = -0.4576 \text{ dB using the + sign} & \text{(Antenna is LH in receiving mode and RH in transmitting)} \\ \frac{1}{10} = -10 \text{ dB using the − sign} & \text{(Antenna is RH in receiving mode and LH in transmitting)} \end{cases}$$

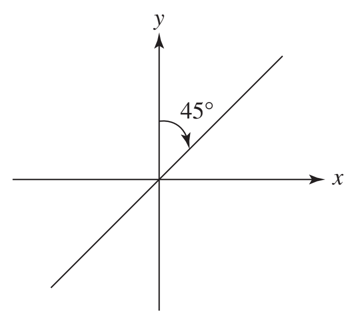
(b) $\hat{\rho}_w = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right) \Rightarrow$ Wave is Left Hand (LH)

$\hat{\rho}_a = \left(\frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}\right)$

$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2$

$$= \begin{cases} \frac{1}{10} = -10 \text{ dB using the + sign} & \text{(Antenna is LH in receiving mode and RH in transmitting)} \\ \frac{9}{10} = -0.4576 \text{ dB using the − sign} & \text{(Antenna is RH in receiving mode and LH in transmitting)} \end{cases}$$

2.44. For $\hat{\rho}_w$



$$\hat{\rho}_w = \frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}}; \;\; \text{PLF} = \left| \frac{\hat{a}_x + \hat{a}_y}{\sqrt{2}} \cdot \frac{4\hat{a}_x + j\hat{a}_y}{\sqrt{17}} \right|^2$$

$$\text{PLF} = \frac{1}{34} |(\hat{a}_x \cdot 4\hat{a}_x) + (\hat{a}_y \cdot j\hat{a}_y)|^2 = \frac{1}{34} |4 + j|^2 = 0.5 \text{ dimensionless} = -3 \text{ dB}$$

2.45. (a) RHCP; $\hat{\rho}_a = \dfrac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}$

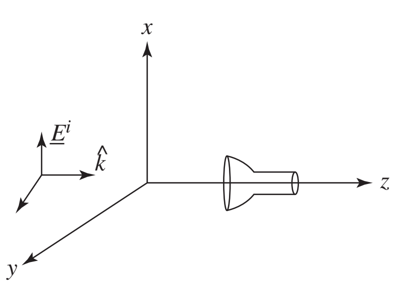
$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left| \frac{2\hat{a}_x + j\hat{a}_y}{\sqrt{5}} \cdot \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}} \right|^2 = 0.9 \text{ dimensionless} = -0.46 \text{ dB}$$

(b) LHCP; $\hat{\rho}_a = \dfrac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left| \frac{2\hat{a}_x + j\hat{a}_y}{\sqrt{5}} \cdot \frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}} \right|^2 = 0.1 \text{ dimensionless} = -10.0 \text{ dB}$$

2.46. $\underline{E}^i = (\hat{a}_x - j\hat{a}_y) E_0 e^{-jkz} = \left(\dfrac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}} \right) \sqrt{2} E_0 e^{-jkz}$

$$\hat{\rho}_w = \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}} \qquad \text{CW}$$



(a) $\underline{E}^a = (\hat{a}_x + j\hat{a}_y)E_1 e^{+jkz}$

$$= \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_1 e^{+jkz}$$

$$\hat{\rho}_a = \frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}} \quad \text{CW}$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right) \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left(\frac{1-j^2}{2}\right)^2 = 1$$

PLF = 1 = 0 dB

(b) $\underline{E}^a = \left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_1 e^{+jkz}$

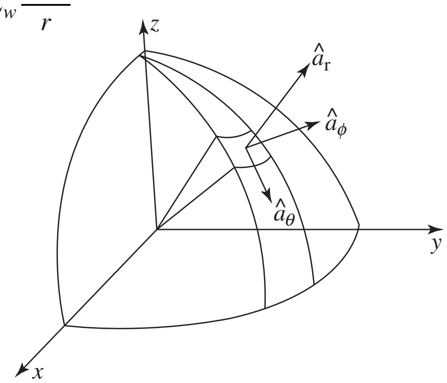
$$\hat{\rho}_a = \frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}$$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right) \cdot \left(\frac{\hat{a}_x - j\hat{a}_y}{\sqrt{2}}\right)\right|^2 = \left|\frac{1+j^2}{2}\right|^2 = 0$$

PLF = 0 = $-\infty$ dB

2.47. $\underline{E}_a = (2\hat{a}_\theta + j4\hat{a}_\phi)E_a \frac{e^{-jkr}}{r} = \underbrace{\left(\frac{2\hat{a}_\theta + j4\hat{a}_\phi}{\sqrt{20}}\right)}_{\hat{\rho}_a} 20E_a \frac{e^{-jkr}}{r}$

$$\underline{E}_w = (j4\hat{a}_\theta + 2\hat{a}_\phi)E_w \frac{e^{+jkr}}{r} = \underbrace{\left(\frac{j4\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{20}}\right)}_{\hat{\rho}_w} 20E_w \frac{e^{+jkr}}{r}$$



Antenna

a. Elliptical
b. CCW
c. **AR** = $\frac{4}{2}$ = 2

Wave

d. Elliptical
e. CCW
f. **AR** = $\frac{4}{2}$ = 2
g.
$$\text{PLF} = |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \left|\left(\frac{2\hat{a}_\theta + j4\hat{a}_\phi}{\sqrt{20}}\right) \cdot \left(\frac{j4\hat{a}_\theta + 2\hat{a}_\phi}{\sqrt{20}}\right)\right|^2$$

$$= \left|\frac{j8 + j8}{20}\right| = \left|\frac{16}{20}\right|^2 = (0.8)^2 = 0.64$$

PLF = 0.64 dimensionless = −1.9382 dB

2.48. $\underline{E}^i = \hat{a}_x E_0 e^{-jkz}, \hat{\rho}_w = \hat{a}_x$

$$\underline{E}^a = (\hat{a}_x + j\hat{a}_y)E_1 e^{+jkz} = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\sqrt{2}E_1 e^{+jkz}$$

$$\hat{\rho}_a = \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)$$

(a) $A_{em} = \dfrac{\lambda^2}{4\pi} e_o D_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \dfrac{\lambda^2}{4\pi} G_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2$

$(e_o D_0 = G_0)$

$$\text{At 10 GHz} \Rightarrow \lambda = \frac{c}{f} = \frac{3\times 10^8}{10\times 10^9} = \frac{3\times 10^8}{10^{10}} = 3\times 10^{-2}$$

$$G_0 = 10 = 10\log_{10} G_0(\text{dim}) \Rightarrow G_0(\text{dim}) = 10^1 = 10$$

$$A_{em} = \frac{\lambda^2}{4\pi} G_0 |\hat{\rho}_a \cdot \hat{\rho}_w|^2 = \frac{(3\times 10^{-2})^2}{4\pi}(10)\left|\hat{a}_x \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}}\right)\right|^2$$

$$= \frac{9\times 10^{-4}}{4\pi}(10)\left(\frac{1}{2}\right) = \frac{9\times 10^{-3}}{4\pi}\left(\frac{1}{2}\right) = (0.7162\times 10^{-3})\left(\frac{1}{2}\right)$$

$$A_{em} = 0.3581\times 10^{-3}\ \text{m}^2$$

(b) $P_T = A_{em} W^i = (0.3581\times 10^{-3})(10\times 10^{-3}) = 3.581\times 10^{-6}$ Watts

$P_T = 3.581\times 10^{-6}$ Watts

2.49. $\underline{E}_w = \hat{a}_z E_w \dfrac{e^{+jky}}{y}, \hat{\rho}_w = \hat{a}_z, E_a = -\hat{a}_z E_a \dfrac{e^{-jky}}{y}, \hat{\rho}_a = -\hat{a}_z, W_{\text{inc}} = 100\times 10^{-3} \dfrac{W}{\text{cm}^2}$

(a) $\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = |-\hat{a}_z \cdot \hat{a}_z|^2 = 1 = 0$ dB

(b) For the $\lambda/2$ dipole ($Z_a = 73 + j42.5$) with a loss resistance R_L of 5 ohms:

$$U_n = (E_{\theta n})^2 = (\sin^{1.3}\theta)^2 = \sin^3\theta \Rightarrow (U_n)_{\max} = 1$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\left(\int_0^{\pi}\sin^3\theta\sin\theta\, d\theta\right) d\phi = 2\pi\int_0^{\pi}\sin^4\theta\, d\theta$$

$$\int_0^{\pi}\sin^4\theta\, d\theta = -\frac{\sin^3\theta\cos\theta}{4}\bigg|_0^{\pi} + \frac{4-1}{4}\int_0^{\pi}\sin^2\theta\, d\theta = \frac{3}{4}\int_0^{\pi}\sin^2\theta\, d\theta$$

$$\int_0^{\pi}\sin^4\theta\, d\theta = \frac{3}{4}\int_0^{\pi}\sin^2\theta\, d\theta = \frac{3}{4}\left[\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right] = \frac{3\pi}{8}$$

$$P_{\text{rad}} = 2\pi \int_0^{\pi} \sin^4 \theta d\theta = 2\pi \left(\frac{3\pi}{8}\right) = \frac{3\pi^2}{4}$$

$$\therefore D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = D_0 = \frac{4\pi(1)}{3\pi^2/4} = \frac{16}{3\pi} = 1.69765 \text{ (dimensionless)} = 2.298 \text{ dB}$$

Using the equivalent circuit of Figure 1.2 with $R_r = 73$ and $R_L = 5$

$$e_{cd} = \frac{R_r}{R_r + R_L} = \frac{73}{73+5} = 0.9359$$

$$\therefore G_0 = e_{cd} D_0 = 0.9359(1.69765) = 1.5888 = 2.011 \text{ dB}$$

$$|\Gamma| = \left|\frac{Z_{in} - Z_c}{Z_{in} + Z_c}\right| = \left|\frac{(Z_a + R_L) - Z_c}{(Z_a + R_L) + Z_c}\right| = \left|\frac{(73 + j42.5 + 5) - 50}{(73 + j42.5 + 5) + 50}\right| = \frac{50.8945}{134.8712} = 0.3774$$

$$|\Gamma|^2 = (0.3774)^2 = 0.1424 \Rightarrow \left(1 - |\Gamma|^2\right) = (1 - 0.1424) = 0.8576$$

$$G_{re0} = e_r G_0 = \left(1 - |\Gamma|^2\right) G_0 = (0.8576)\, 1.5888 = 1.3626 \text{ (dim)} = 1.344 \text{ dB}$$

$$P_{\text{received}} = A_{em}(e_{cd})(1 - |\Gamma|^2)\text{PLF} = \left(\frac{\lambda^2}{4\pi} D_0\right) e_{cd}(1 - |\Gamma|^2)\text{PLF}$$

$$= \frac{\pi^2}{4\pi}\underbrace{\left[\underbrace{D_0 e_{cd}}_{G_0}(1 - |\Gamma|^2)\right]}_{G_{re0}}(\text{PLF})W_{inc}$$

$$P_{\text{received}} = (W_{inc})\frac{\lambda^2}{4\pi} G_{re}(\text{PLF}), \qquad \lambda = \frac{30 \times 10^9}{10 \times 10^9} = 3 \text{ cm}$$

$$= \underbrace{(100 \times 10^{-3})}_{W_{inc}} \underbrace{\left(\frac{(3)^2}{4\pi}\right)}_{3^2/4\pi} \underbrace{(1.3562)}_{G_{re0}} \underbrace{(1)}_{\text{PLF}} = \frac{10^{-1}(9)(1.3562)}{4\pi}$$

$$P_{\text{received}} = 0.0981 \text{ Watts} = 98.1 \text{ mWatts} = 98.1 \times 10^{-3} \text{ Watts}$$

$$P_{\text{received}} = \underbrace{(100 \times 10^{-3})}_{W_{\text{inc}}} \underbrace{\left[\frac{(3)^2}{4\pi}\right]}_{3^2/4\pi} \underbrace{(1.3626)}_{G_{re0}} \underbrace{(1)}_{\text{PLF}} = 97.59 \text{ mW} = 97.59 \times 10^{-3} \text{ W}$$

2.50. $\underline{E}_a = (2\hat{a}_x \pm j\hat{a}_y)Ee^{-jkz}$

$$\hat{\rho}_a = \frac{2\hat{a}_x \pm j\hat{a}_y}{\sqrt{5}}$$

(a) $\underline{E}_w = \hat{a}_x E_w \Rightarrow \hat{\rho}_w = \hat{a}_x$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2}{\sqrt{5}}\right|^2 = \frac{4}{5} = 0.8 \text{ dimensionless} = -0.9691 \text{ dB}$$

(b) $\underline{E}_w = \hat{a}_y E_w \Rightarrow \hat{\rho}_w = \hat{a}_y$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{1}{\sqrt{5}}\right|^2 = \frac{1}{5} = 0.2 \text{ dimensionless} = -6.9897 \text{ dB}$$

2.51. (a) $E_y = E_y' + E_y'' = 3\cos\omega t + 2\cos\omega t = 5\cos\omega t$

$$E_x = E_x' + E_x'' = 7\cos\left(\omega t + \frac{\pi}{2}\right) + 3\cos\left(\omega t - \frac{\pi}{2}\right)$$
$$= -7\sin\omega t + 3\sin\omega t = -4\sin\omega t$$
$$\text{AR} = \frac{5}{4} = 1.25$$

(b) At $\omega t = 0, \underline{E} = 5\hat{a}_y$
At $\omega t = \pi/2 \Rightarrow \underline{E} = -4\hat{a}_x \Rightarrow$ Rotation in CCW

2.52. (a) $\text{PLF} = \frac{1}{2}$ independent of $\psi \rightarrow$ must have CP

$\therefore$ AR = 1.

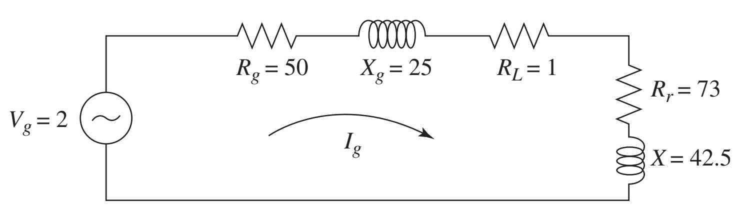
(b) Polarization will be elliptical with major axis aligned with x-axis.
Guess: AR = 2
Verify: $\hat{\rho}_w = (2\hat{a}_x + ja_y)/\sqrt{5}$

$$\text{PLF} = |\hat{\rho}_w \cdot \hat{\rho}_a|^2 = \left|\frac{2\cos\psi + j\sin\psi}{\sqrt{5}}\right|^2 = \frac{4\cos^2\psi + \sin^2\psi}{\sqrt{5}}$$

$\psi = 0$: PLF = 0.8
$\psi = 90°$: PLF = 0.2

(c) PLF = 1 at $\psi = 45°$ and 225°
PLF = 0 at $\psi = 135°$ and 315°
Polarization must be linear at an angle of 45°
$\therefore$ AR = ∞

2.53.
$$I_g = \frac{2}{(50 + 1 + 73) + j(25 + 42.5)} = \frac{2}{124 + j67.5}$$
$$= (12.442 - j6.7724) \times 10^{-3} = 14.166 \times 10^{-3} \angle -28.56°$$



(a) $P_s = \frac{1}{2}\text{Re}(V_g \cdot I_g^*) = \text{Re}(12.442 + j6.7724) \times 10^{-3} = 12.442 \times 10^{-3}$ W

(b) $P_r = \frac{1}{2}|I_g|^2 R_r = 7.325 \times 10^{-3}$ W

(c) $P_L = \frac{1}{2}|I_g|^2 R_L = 0.1003 \times 10^{-3}$ W

The remaining supplied power is dissipated as heat in the internal resistor of the generator, or

$$P_g = \frac{1}{2}|I_g|^2 R_g = 5.0169 \times 10^{-3} \text{ W}$$

Thus

$$P_r + P_L + P_g = (7.325 + 0.1003 + 5.0169) \times 10^{-3} = 12.4422 \times 10^{-3} = P_s$$

2.54. The impedance transfer equation of

$$Z_{in} = Z_c \left[\frac{Z_L + jZ_c \tan(kl)}{Z_c + jZ_L \tan(kl)}\right]$$

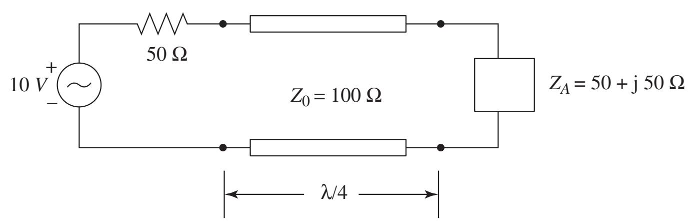
reduces for $l = \lambda/2$ to $Z_{in} = Z_L$.

Therefore the equivalent load impedance at the terminals of the generator is the same as that for Problem 2.53.

Thus the supplied, radiated, and dissipated powers are the same as those of Problem 2.53.

2.55. (a) $Z_{in} = \dfrac{(100)^2}{50 + j50} = \dfrac{10000}{5000}(50 - j50)2 = 100 - j100\ \Omega$

$$I_g = \frac{10}{150 - j100} = \frac{10}{180.3\angle - 33.7°} = 0.05546\angle 33.7°\ A$$



(b) $P_s = \dfrac{1}{2}\text{Re}\{V_g I_g^*\} = \dfrac{1}{2} \times 10 \times 0.05546 \times \cos(33.7°) = 0.231\ \text{W}$

(c) $P_A = \dfrac{1}{2}|I_g|^2 \text{Re}\{Z_{in}\} = \dfrac{1}{2} \times (0.05546)^2 \times 100 = 0.1538\ \text{W}$

$$P_{\text{rad}} = e_{cd} P_A = 0.96 \times 0.1538 = 0.148\ \text{W}$$

2.56.

$$\text{Gain} = \frac{P_{\text{rad}}}{P_{\text{accepted}}}(\text{Directivity})$$

$$\text{Realized Gain} = \frac{P_{\text{rad}}}{P_{\text{available}}}(\text{Directivity})$$

$$\frac{\text{Gain}}{\text{Realized Gain}} = \frac{P_{\text{available}}}{P_{\text{accepted}}}$$

$$P_{\text{available}} = \frac{1}{2}\frac{\left(\frac{V_s}{\sqrt{2}}\right)^2}{Z_0} = \frac{V_s^2}{4Z_0}$$

$$V(x) = A[e^{-jkx} + \Gamma(0)e^{jkx}]$$

$$I(x) = \frac{A}{Z_0}[e^{-jkx} - \Gamma(0)e^{jkx}]$$

$$V(0) = A[1 + \Gamma(0)]$$

$$I(0) = \frac{A}{Z_0}[1 - \Gamma(0)]$$

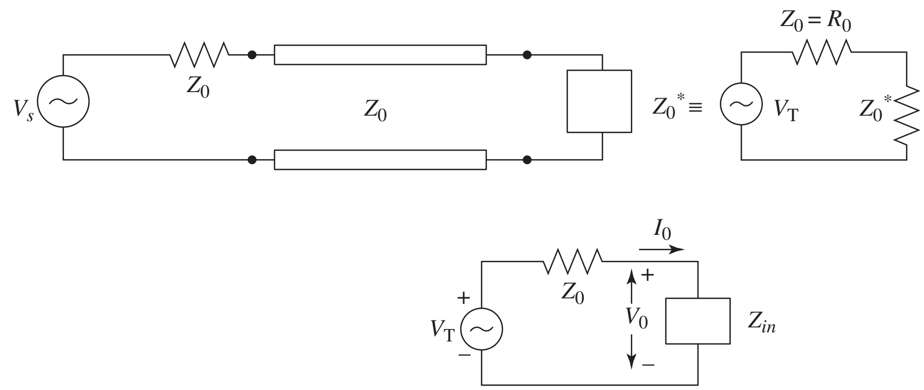


Fig. 1

From Fig. 1:

$$-V_s + I(0)Z_0 + V(0) = 0$$

$$-V_s + \frac{A}{Z_0}[1 - \Gamma(0)]Z_0 + A[1 + \Gamma(0)] = 0$$

$$-V_s + A - A\Gamma(0) + A + A\Gamma(0) = 0$$

$$2A = V_s \rightarrow A = \frac{V_s}{2}$$

$$P_{\text{accepted}} = \text{Re}[V(0)I^*(0)]$$

$$V(0) = \frac{V_s}{2}[1 + \Gamma(0)]$$

$$I(0) = \frac{V_s}{2Z_0}[1 - \Gamma(0)]$$

$$\Gamma(0) = \frac{Z_{in} - Z_0}{Z_{in} + Z_0}$$

$$\Rightarrow V(0) = \frac{V_s}{2}\left(1 + \frac{Z_{in} - Z_0}{Z_{in} + Z_0}\right)$$

$$= \frac{V_s}{2}\left(1 + \frac{R_{in} + jX_{in} - Z_0}{R_{in} + jX_{in} + Z_0}\right)$$

$$= \frac{V_s}{2}\left(\frac{R_{in} + jX_{in} + Z_0 + R_{in} + jX_{in} - Z_0}{R_{in} + jX_{in} + Z_0}\right)$$

$$V(0) = \frac{V_s(R_{in} + jX_{in})}{R_{in} + jX_{in} + Z_0}$$

$$I(0) = \frac{V_s}{2Z_0}\left(1 - \frac{Z_{in} - Z_0}{Z_{in} + Z_0}\right) = \frac{V_s}{2Z_0}\left(\frac{Z_{in} + Z_0 - Z_{in} + Z_0}{Z_{in} + Z_0}\right)$$

$$I(0) = \frac{V_s}{Z_{in} + Z_0} = \frac{V_s}{R_{in} + jX_{in} + Z_0}$$

$$\text{Re}[V(0)I(0)^*] = \text{Re}\left[\frac{V_sR_{in} + jV_sX_{in}}{R_{in} + Z_0 + jX_{in}} \times \frac{V_s}{R_{in} + Z_0 - jX_{in}}\right]$$

$$P_{\text{accepted}} = \text{Re}\left(\frac{V_s^2(R_{in}+jX_{in})}{(R_{in}+Z_0)^2+X_{in}^2}\right) = \frac{V_s^2 R_{in}}{(R_{in}+Z_0)^2+X_{in}^2}$$

$$\frac{\text{Gain}}{\text{Realized Gain}} = \frac{\frac{V_s^2}{4Z_0}}{\frac{V_s^2 R_{in}}{(R_{in}+Z_0)^2+X_{in}^2}} = \frac{(R_{in}+Z_0)^2+X_{in}^2}{4Z_0 R_{in}}$$

2.57. (a) $R_L = R_{hf} = \frac{l}{C}\sqrt{\frac{\omega\mu_0}{2\sigma}}$

$$= \frac{\lambda/60}{2\pi(\lambda/200)} \cdot \sqrt{\frac{2\pi\times 10^9(4\pi\times 10^{-7})}{2(5.7\times 10^7)}}$$

$R_L = 0.4415\times 10^{-2} = 0.004415$ ohms

(b) $R_r = 80\pi^2\left(\frac{l}{\lambda}\right)^2 = 80\pi^2\left(\frac{1}{60}\right)^2 = 0.21932$

$\Rightarrow R_{in} = R_r = 0.21932$ ohms (because of assumed constant current)

(c) $e_{cd} = \frac{R_r}{R_L+R_r} = \frac{0.21932}{0.21932+0.004415} = 0.98027$

$e_{cd} = 98.027\%$

(d) $Z_L = (R_L+R_{in}) + jX_{in} = (0.21932+0.004415)+jX_{in}$

$= 0.2237 + jX_{in}$

$$X_{in} \simeq -120\frac{\ln(l/2a)-1}{\tan(\frac{kl}{2})} = -120\frac{\left[\ln\left(\frac{\lambda/60}{\lambda/100}\right)-1\right]}{\tan\left(\frac{2\pi}{2\lambda}\frac{\lambda}{60}\right)}$$

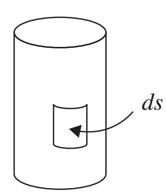
$$= -120\left[\frac{0.51003-1}{0.05241}\right] = +1,120.03$$

$$|\Gamma| = \frac{Z_L-Z_c}{Z_L+Z_c} = \frac{(0.2237+j1,120.03)-50}{(0.2237+j1,120.03)+50} = 0.9999$$

$$\text{VSWR} = \frac{1+|\Gamma|}{1-|\Gamma|} = \frac{1+0.9999}{1-0.9999} = 9,999$$

2.58. Radiation Efficiency of a dipole

$I_z(z) = I_0\cos\left[\frac{\pi}{l}z'\right], -l/2 \le z' \le l/2$



$H_\phi(r=a)|_{\text{at the surface}} = \frac{I_0}{2\pi a}\cos\left[\frac{\pi}{l}z\right]$

$ds = a\,d\phi\,dz \Rightarrow$ differential patch of area.

$dW \Rightarrow$ power loss into this patch.

$dW = \frac{1}{2}|H_\phi|^2 R_s a\,d\phi\,dz$

(time ave) (R_s = skin resistance)

$$dW = \left(\frac{I_0}{2\pi a}\right)^2 \cdot \frac{R_s}{2}\cos^2\left[\frac{\pi}{l}z\right] a\,d\phi\,dz$$

$$W(\text{total loss}) = \int_{-l/2}^{l/2}\int_{\phi=0}^{2\pi} \frac{{I_0}^2 R_s}{8\pi^2 \cdot a^2} \cos^2\left[\frac{\pi}{l}z\right]\, a\, d\phi\, dz$$

$$W = \frac{{I_0}^2}{8\pi^2 a^2}(2\pi a)R_s \int_{-l/2}^{l/2} \cos^2\left[\frac{\pi}{l}z\right] dz = \frac{{I_0}^2}{4\pi}\frac{l \cdot R_s}{a}\left(\frac{1}{2}\right)$$

$$W = \frac{1}{2}{I_0}^2 RL$$

$$R_L = \frac{1}{2}\left(\frac{lR_s}{2\pi a}\right)$$

2.59. $E = \begin{cases} 1 & 0 < \theta \leq 45^\circ \\ 0 & 45^\circ < \theta \leq 90^\circ \\ \frac{1}{2} & 90^\circ < \theta \leq 180^\circ \end{cases}$

(a) $U = \dfrac{r^2 E^2}{2\eta} = \dfrac{r^2|E|^2}{\eta}, \qquad U_{\text{max}} = \dfrac{r^2}{\eta} = \dfrac{1}{120\pi}$

$$P_{\text{rad}} = \frac{r^2}{\eta}\int_0^{2\pi} d\phi \left[\int_0^{45^\circ} \sin\theta\, d\theta + \int_{90^\circ}^{180^\circ} \frac{1}{4}\sin\theta\, d\theta\right]$$

$$= \frac{r^2}{\eta}[2\pi]\left[-\cos\theta|_0^{45^\circ} + \frac{1}{4}(-\cos\theta)|_{90^\circ}^{180^\circ}\right]$$

$$= \frac{2r^2\pi}{\eta}\left[-\cos 45^\circ + \cos 0^\circ - \frac{1}{4}\cos 180^\circ + \frac{1}{4}\cos 90^\circ\right]$$

$$P_{\text{rad}} = 0.54289\frac{2\pi r^2}{\eta}$$

$$D = \frac{4\pi U_{\text{max}}}{P_{\text{rad}}} = \frac{4\pi\left(\dfrac{r^2}{\eta}\right)}{0.54289(2\pi)r^2/\eta} = 3.684$$

(b) When the electric field is equal to 10 V/m, for $\theta = 0^\circ$.

$$\Rightarrow E = \begin{cases} 10\text{ V/m} & 0 < \theta \leq 45^\circ \\ 0 & 45^\circ < \theta \leq 90^\circ \\ \frac{1}{2} \times 10\text{ V/m} & 90^\circ < \theta \leq 180^\circ \end{cases}$$

$$P_{\text{rad}} = \frac{r^2}{\eta}\left[\int_0^{2\pi}\left\{\int_0^{45^\circ} |E|^2 \sin\theta\, d\theta + \int_{90^\circ}^{180^\circ} |E|^2 \sin\theta\, d\theta\right\} d\phi\right]$$

$$P_{\text{rad}} = r^2(0.54289)\left(\frac{2\pi}{\eta}\right)|10|^2 = 36{,}193$$

$$P_{\text{rad}} = \frac{1}{2}|I|^2 R_r = |I_{\text{rms}}|^2 \cdot R_r$$

$$\Rightarrow R_r = \frac{36{,}193}{|I_{\text{rms}}|^2} = \frac{36{,}193}{25} = 1{,}447.72$$

2.60. $\underline{E}_a = \hat{a}_\theta E_a \cos^3\theta \dfrac{e^{-jkr}}{r} \Rightarrow U_n = (\cos^3\theta)^2 = \cos^6\theta$

(a) $U_n|_{max} = \cos^6\theta|_{max} = 1, \theta_{max} = 0°$

$$U_n|_{\theta=\theta_h} = \cos^6\theta_h = 0.5 \Rightarrow \theta_h = \cos^{-1}[(0.5)^{1/6}] = \cos^{-1}(0.891) = 27.01°$$

$$\Theta_h = \text{HPBW} = 2\theta_h = 2(27.01) = 54.02°$$

(b) **Exact Directivity:**

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} U_n \sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\int_0^{\pi/2} \cos^6\theta \sin\theta\, d\theta\, d\phi = 2\pi\int_0^{\pi/2}\cos^6\theta\, \sin\theta\, d\theta$$

$$= -2\pi\int_0^{\pi/2}(\cos\theta)^6 d(\cos\theta) = -2\pi\left(\frac{\cos^7\theta}{7}\right)_0^{\pi/2} = -2\pi\left(0-\frac{1}{7}\right) = 2\pi/7$$

$$D_0 = \frac{4\pi U_{max}}{P_{rad}} = \frac{4\pi(1)}{2\pi/7} = 14 = 10\log_{10}(14) = 11.46 \text{ dB}$$

(c) Since the HPBW = 54.02° > 39.77°($n = 6 < 11.48$), then Kraus' approximate formula is the more accurate for the maximum directivity. Thus

$$D_0(\text{Kraus}) = \left.\frac{41{,}253}{\Theta_{1h}\Theta_{2h}}\right|_{\Theta_{1h}=\Theta_{2h}} = \frac{41{,}253}{\Theta_h^2} = \frac{41{,}253}{(54.02)^2} = 14.137$$

$$D_0(\text{Kraus}) = 14.137 \text{ (dimensionless)} = 10\log_{10}(14.137) = 11.50 \text{ dB}$$

$$D_0(\text{Tai \& Pereira}) = \left.\frac{72{,}815}{\Theta_{1h}^2+\Theta_{2h}^2}\right|_{\Theta_{1h}=\Theta_{2h}} = \frac{72{,}815}{2\Theta_h^2} = \frac{72{,}815}{2(54.02)^2} = 12.476$$

$$D_0(\text{T \& P}) = 12.476 \text{ (dimensionless)} = 10\log_{10}(12.476) = 10.961 \text{ dB}$$

By comparsion, the Kraus' approximate formula D_0 is more accurate, compared to the exact D_0, for this problem.

(d) Using the computer program **Directivity**, the maximum directivity is

$$D_0 = 13.9637 \text{ (dimensionless)} = 11.45 \text{ dB}$$

Basically identical to the exact value.

(e) $A_{em} = \dfrac{\lambda^2}{4\pi} D_0|_{exact} = \dfrac{\lambda^2}{4\pi}(14) = 1.141\lambda^2$

```
Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
-------------------
Radiated power (watts) = 0.8999
Directivity (dimensionless) = 13.9637
Directivity (dB) = 11.4500
```

2.61. In general,

$$D_{\theta o} = \frac{4\pi (U_\theta)_{\max}}{(P_{\text{rad}})_\theta + (P_{\text{rad}})_\phi}; \quad D_{\phi o} = \frac{4\pi (U_\phi)_{\max}}{(P_{\text{rad}})_\theta + (P_{\text{rad}})_\phi}$$

$$U_\theta = |E_\theta|^2; \quad U_\phi = |E_\phi|^2$$

$$U = U_\theta + U_\phi = |E|^2 = |E_\theta|^2 + |E_\phi|^2$$

However for this problem

$$U_{\max}(\theta = 0^\circ; \phi = 0^\circ \text{ or } 90^\circ \text{ or any value } 0 \le \phi \le 2\pi) = |E|^2_{\max} = |E_\theta|^2_{\max} = |E_\phi|^2_{\max}$$

$$(P_{\text{rad}})_\theta = \int_0^{2\pi}\int_0^{\pi/2} U_\theta \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi/2} |E_\theta|^2 \sin\theta \, d\theta \, d\phi$$

$$(P_{\text{rad}})_\phi = \int_0^{2\pi}\int_0^{\pi/2} U_\phi \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi/2} |E_\phi|^2 \sin\theta \, d\theta \, d\phi$$

$$P_{\text{rad}} = (P_{\text{rad}})_\theta + (P_{\text{rad}})_\phi = \int_0^{2\pi}\int_0^{\pi/2} U \sin\theta \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{\pi/2} \left[U_\theta + U_\phi\right] \sin\theta \, d\theta \, d\phi$$

$$P_{\text{rad}} = (P_{\text{rad}})_\theta + (P_{\text{rad}})_\phi = \int_0^{2\pi}\int_0^{\pi/2} \left[|E_\theta|^2 + |E_\phi|^2\right] \sin\theta \, d\theta \, d\phi$$

However, since for this problem

$$U_{\max}(\theta = 0^\circ; \phi = 0^\circ \text{or } 90^\circ \text{or any value } 0 \le \phi \le 2\pi) = |E|^2_{\max} = |E_\theta|^2_{\max} = |E_\phi|^2_{\max}$$

$$D_0 = D_{\theta 0} = D_{\phi 0}; \text{ NOT } D_0 = D_{\theta 0} + D_{\phi 0}$$

However, in general, for any problem, other than special cases like Problem 2.61

$$D_0 = D_{\theta 0} + D_{\phi 0}$$

if $U_{\max} = |E|^2_{\max} = |E_\theta|^2_{\max} + |E_\phi|^2_{\max} \ne |E_\theta|^2_{\max} \ne |E_\phi|^2_{\max}$

```
Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
-------------------
Radiated power (watts) = 0.1566
Partial Directivity (theta) (dimensionless) = 80.2511
Partial Directivity (theta) (dB) = 19.0445
Partial Directivity (phi) (dimensionless) = 80.2511
Partial Directivity (phi) (dB) = 19.0445
Directivity (dimensionless) = 80.2511
Directivity (dB) = 19.0445
```

Using Table 12.1

$$a = 3\lambda, b = 2\lambda$$

$$D_0 = 4\pi\left(\frac{ab}{\lambda^2}\right) = 4\pi(6) = 24\pi$$

$$D_0 = 75.398 = 18.774 \text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$ then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2.62.

```
Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
------------------
Radiated power (watts) = 0.0330
Partial Directivity (theta) (dimensionless) = 62.4635
Partial Directivity (theta) (dB) = 17.9563
Partial Directivity (phi) (dimensionless) = 62.4635
Partial Directivity (phi) (dB) = 17.9563
Directivity (dimensionless) = 62.4635
Directivity (dB) = 17.9563
```

Using Table 12.1

$$a = 3\lambda, b = 2\lambda$$

$$D_0 = 0.81\left(4\pi\frac{ab}{\lambda^2}\right) = 0.81(24\pi)$$

$$= 61.072 = 17.858 \text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2.63.

```
Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
------------------
Radiated power (watts) = 0.4863
Partial Directivity (theta) (dimensionless) = 4.2443
Partial Directivity (theta) (dB) = 6.2780
```

```
Partial Directivity (phi) (dimensionless) = 4.2443
Partial Directivity (phi) (dB) = 6.2780
Directivity (dimensionless) = 4.2443
Directivity (dB) = 6.2780
```

Using Table 12.1

$$f = 10 \text{ GHz} \Rightarrow \lambda = 3 \text{ cm} \Rightarrow a = \frac{2.286}{3}\lambda = 0.762\lambda$$

$$b = \frac{1.016}{3}\lambda = 0.3387\lambda$$

$$D_0 = 0.81\left(4\pi\frac{ab}{\lambda^2}\right) = 0.81(4\pi)(0.762)(0.3387)$$

$$= 2.627 = 4.194 \text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2.64.

```
Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
------------------
Radiated power (watts) = 0.0338
Partial Directivity (theta) (dimensionless) = 92.9470
Partial Directivity (theta) (dB) = 19.6824
Partial Directivity (phi) (dimensionless) = 92.9470
Partial Directivity (phi) (dB) = 19.6824
Directivity (dimensionless) = 92.9470
Directivity (dB) = 19.6824
```

Using Table 12.2

$$a = 1.5\lambda$$

$$D_0 = \frac{4\pi}{\lambda^2}(\pi a^2) = \left(\frac{2\pi a}{\lambda}\right)^2 = 9\pi^2$$

$$D_0 = 88.826 = 19.485 \text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2.65.

```
Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360
```

```
Output parameters:
------------------
Radiated power (watts) = 0.0418
Partial Directivity (theta) (dimensionless) = 75.1735
Partial Directivity (theta) (dB) = 18.7606
Partial Directivity (phi) (dimensionless) = 75.1735
Partial Directivity (phi) (dB) = 18.7606
Directivity (dimensionless) = 75.1735
Directivity (dB) = 18.7606
```

Using Table 12.2

$$a = 1.5\lambda$$

$$D_0 = 0.836\left(\frac{2\pi a}{\lambda}\right)^2 = 0.836\,(9\pi^2)$$

$$D_0 = 74.2589 = 18.71\text{ dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2.66.

```
Input parameters:
-----------------
The lower bound of theta in degrees = 0
The upper bound of theta in degrees = 90
The lower bound of phi in degrees = 0
The upper bound of phi in degrees = 360

Output parameters:
------------------
Radiated power (watts) = 0.4952
Partial Directivity (theta) (dimensionless) = 6.3439
Partial Directivity (theta) (dB) = 8.0236
Partial Directivity (phi) (dimensionless) = 6.3439
Partial Directivity (phi) (dB) = 8.0236
Directivity (dimensionless) = 6.3439
Directivity (dB) = 8.0236
```

Using Table 12.2

$$f = 10\,\text{GHZ} \Rightarrow \lambda = 3\,\text{cm} \Rightarrow a = \frac{1.143}{3}\lambda = 0.381\lambda$$

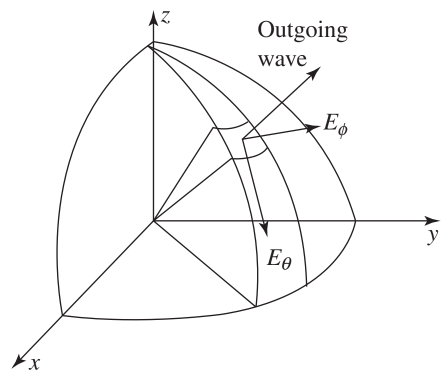
$$D_0 = 0.836\left(\frac{2\pi a}{\lambda}\right)^2 = 0.836[2\pi(0.381)]^2$$

$$D_0 = 4.791 = 6.804\,\text{dB}$$

Since the maximum $|E_\theta| = |E_\phi| = |\underline{E}|$, then the maximum directivity

$$D_0 = D_\theta = D_\phi$$

2.67. $\underline{E}_a = (\hat{a}_\theta + j2\hat{a}_\phi)\ \sin\theta\, E_0 \dfrac{e^{-jkr}}{r}\ (0° \le \theta \le 180°,\ 0° \le \phi \le 360°)$



(a) Elliptical because
1. 2 components transverse to the direction of wave propagation
2. Both components *not* of the same magnitude
3. 90° phase difference between the 2 components
4. AR = 2/1 = 2 because ellipse aligned with principal axes
5. CCW (E_ϕ leads E_θ) due to the 90° phase difference between the two.

(b) $(D_0)_\theta = \dfrac{4\pi(U_\theta)_{\max}}{(P_{\text{rad}})_\theta + (P_{\text{rad}})_\phi}; \qquad (D_0)_\phi = \dfrac{4\pi(U_\phi)_{\max}}{(P_{\text{rad}})_\theta + (P_{\text{rad}})_\phi}$

$$(U_t)_n = (U_\theta)_n + (U_\phi)_n = |E_\theta|_n^2 + |E_\phi|_n^2 = (1+4)\sin^2\theta|E_0|^2 = 5\sin^2\theta|E_0|^2$$

$$(U_t)_n = 5\sin^2\theta|E_0|^2$$

$$(U_\theta)_n = |E_\theta|_n^2 = \sin^2\theta|E_0|^2; \quad (U_\phi)_n = |E_\phi|_n^2 = 4\sin^2\theta|E_0|^2$$

$$(U_\theta)_{n\max} = |E_0|^2; \quad (U_\phi)_{\text{nmax}} = 4|E_0|^2$$

$$(P_{\text{rad}})_t = (P_{\text{rad}})_\theta + (P_{\text{rad}})_\phi = \int_0^{2\pi}\int_0^{\pi} (U_t)_n \sin\theta\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi} 5\sin^2\theta|E_0|^2 \sin\theta\, d\theta\, d\phi = 5|E_0|^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta\, d\theta$$

$$= 5(2\pi)|E_0|^2 \int_0^{\pi} \sin^3\theta\, d\theta = 10\pi|E_0|^2\left(\frac{4}{3}\right) = \frac{40\pi}{3}|E_0|^2$$

$$(P_{\text{rad}})_t = \frac{40\pi}{3}|E_0|^2$$

$$(D_0)_\theta = \frac{4\pi(U_\theta)_{\max}}{(P_{\text{rad}})_t} = \frac{4\pi|E_0|^2}{\frac{40\pi}{3}|E_0|^2} = \frac{3}{10} = \boxed{0.3 = -5.2288 \text{ dB}}$$

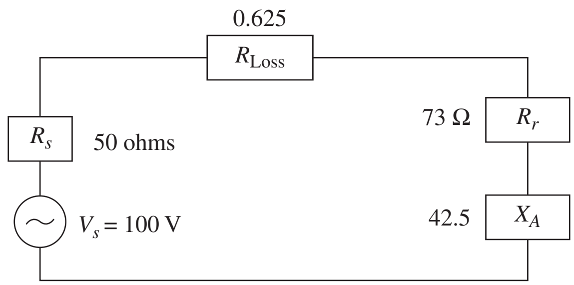
$$(D_0)_\phi = \frac{4\pi(U_\phi)_{\max}}{(P_{\text{rad}})_t} = \frac{4\pi(4)|E_0|^2}{\frac{40\pi}{3}|E_0|^2} = \frac{12}{10} = \boxed{1.2 = 0.79181 \text{ dB}}$$

(c) $(D_0)_t = (D_0)_\theta + (D_0)_\phi = 0.3 + 1.2 = \boxed{1.5 = 1.761 \text{ dB}}$

2.68. $f = 150$ MHz, $\lambda = 2$ m

$\Rightarrow$ 1 m dipole is $\frac{\lambda}{2}$ in electrical length

$\Rightarrow R_r = 73$ ohms, $Z_{in} = 73 + j42.5$ ohms [see (4-93), Chapter 4]



(a) $I_{\text{ant}} = \dfrac{V_s}{50 + 73 + 0.625 + j42.5} = 0.765\angle - 18.97^\circ\ A$

(b) $P_{\text{dissip}} = P_{\text{Loss}} = \dfrac{1}{2}|I_{\text{ant}}|^2 \cdot R_{\text{Loss}} = 189\,\text{mW}$

(c) $P_{\text{rad}} = \dfrac{1}{2}|I_{\text{ant}}|^2 \cdot R_r = 21.36\ W$

(d) $E_{cd} = \dfrac{R_r}{R_r + R_{\text{Loss}}} = \dfrac{73}{73 + 0.625} = 99\%$

2.69. $\underline{E} = \hat{a}_\theta E_\theta \simeq \hat{a}_\theta j\eta \dfrac{kI_0 l}{4\pi r} e^{-jkr} \sin\theta = -j\eta \dfrac{kI_0 e^{-jkr}}{4\pi r}[\underbrace{-\hat{a}_\theta l \sin\theta}_{\underline{l}_e}]$

(a) $\underline{l}_e = -\hat{a}_\theta l \sin\theta$

(b) $|l_e|_{\max} = |-\hat{a}_\theta l \sin\theta|_{\max} = l$ @ $\theta = 90^\circ$

(c) $|l_e|_{\max}/l = 1$

2.70.

$$\underline{E} = \hat{a}_\theta E_\theta = \hat{a}_\theta j\eta \frac{I_0 e^{-jkr}}{2\pi r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]$$

$$= j\eta \frac{kI_0 e^{-jkr}}{4\pi r}\left[-\hat{a}_\theta \frac{2\cos\left(\frac{\pi}{2}\cos\theta\right)}{k\sin\theta}\right]$$

$$= j\eta \frac{kI_0 e^{-jkr}}{4\pi r}\left[\underbrace{-\hat{a}_\theta \frac{\lambda}{\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}}_{\underline{l}_e}\right]$$

$$\underline{l}_e = -\hat{a}_\theta \frac{\lambda}{\pi}\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} = -\hat{a}_\theta 0.3183\lambda \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$|\underline{l}_e|_{\max} = \left| -\hat{a}_\theta 0.3183\lambda \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \right|_{\max} = 0.3183\lambda \text{ @ } \theta = 90^\circ$$

$$\frac{|l_e|_{\max}}{l} = \frac{0.3183\lambda}{\lambda/2} = 0.6366 = 63.66\% \text{ @ } \theta = 90^\circ$$

2.71.

$$\underline{l}_e = -\hat{a}_\theta l \sin\theta, l = \lambda/50, f = 10 \text{ GHz} \Rightarrow \lambda = 3 \text{ cm}$$

$$W = \frac{1}{2\eta}|\underline{E}|^2 = 10^{-3} \text{ W/cm} \Rightarrow |E| = \sqrt{2\eta W}$$

$$= \sqrt{2(377)(10^{-3})} = 0.8683 \text{ V/cm}$$

$$V_{oc}|_{\max} = |\underline{E}^i||\underline{l}_e|_{\max} = (0.8683)\left(\frac{\lambda}{50}\right) = 52.1 \times 10^{-3} \text{ Volts}$$

2.72. Since $|\underline{l}_e|_{\max} = l/2 \Rightarrow |V_{oc}|_{\max} = \frac{1}{2}$ (V_{oc} of dipole with uniform current)
$V_{oc}|_{\max} = \frac{1}{2}(52.1 \times 10^{-3}) = 26.05 \times 10^{-3}$ Volts (see Problem 2.71)

2.73. $|\underline{l}_e|_{\max} = 0.3183\lambda \Rightarrow |V_{oc}| = |l_e|_{\max}|E^i|$. From Problem 2.71 solution
$|V_{oc}| = 0.8683(0.3183\lambda) = 0.27638\lambda = 0.27638(3) = 0.82914$ Volts

2.74. Using (2.94), the effective aperture of an atenna can be written as

$$A_e = \frac{|V_T|^2 \cdot R_T}{2W_i|Z_t|^2}, \text{ where } W_i = |E|^2/2\eta$$

Defining the effective length l_e as $V_T = \underline{E} \cdot \underline{l}_e$ reduces A_e to

$$A_e = \frac{\eta R_T l_e^2}{|Z_t|^2} \Rightarrow l_e = \sqrt{\frac{A_e|Z_t|^2}{\eta R_T}}$$

For maximum power transfer and lossess antenna ($R_L = 0$)

$$X_A = -X_T, R_r = R_T \Rightarrow |Z_t| = 2R_r = 2R_T$$

$$\text{Thus } l_e = \sqrt{\frac{4A_{em} \cdot R_T^2}{\eta R_T}} = 2\sqrt{\frac{A_{em}R_T}{\eta}} = 2\sqrt{\frac{A_{em}R_r}{\eta}}$$

2.75.

$$A_{em} = 2.147 = \left(\frac{\lambda^2}{4\pi}\right) \cdot e_{cd} \cdot (1 - |\Gamma|^2) \cdot |\hat{\rho}_w \cdot \hat{\rho}_a|^2 \cdot D_0$$

$$\Gamma = \frac{75 - 50}{75 + 50} = 0.2; \; \lambda = \frac{3 \times 10^8}{100 \times 10^6} = 3 \text{ m}$$

$$\therefore D_0 = \frac{2.147}{\frac{3^2}{4\pi}[(1 - (0.2)^2]} = 3.125$$

2.76. $d = 1$ m, $f = 3$ GHz, $\varepsilon_{ap} = 68\% \Rightarrow \lambda = \dfrac{3 \times 10^8}{3 \times 10^9} = 0.1$ m

(a) $A_p = \pi r^2 = \pi\left(\dfrac{d}{2}\right)^2 = \dfrac{\pi\, d^2}{4} = \dfrac{\pi(1)^2}{4} = \boxed{\frac{\pi}{4} = 0.785 \text{ m}^2}$

(b) $\varepsilon_{ap} = \dfrac{A_{em}}{A_p} \Rightarrow A_{em} = \varepsilon_{ap} A_p$

$$A_{em} = \varepsilon_{ap} A_p = 0.68(0.785) = \boxed{0.534 \text{ m}^2}$$

(c) $A_{em} = \dfrac{\lambda^2}{4\pi} D_0 \Rightarrow D_0 = \dfrac{4\pi}{\lambda^2} A_{em}$

$$D_0 = \frac{4\pi}{\lambda^2} A_{em} = \frac{4\pi}{(0.1)^2}(0.534) = \frac{4\pi}{0.01}(0.534) = 671.044$$

$$D_0 = \boxed{671.044 = 28.268 \text{ dB}}$$

(d) $P_L = A_{em} W_i = 0.534(10 \times 10^{-6})$

$$P_L = \boxed{5.34 \times 10^{-6} \text{ Watts}}$$

2.77. $W_i = 10 \times 10^{-3}\ \text{W/cm}^2, \quad l = \lambda/2, \quad D_0 = 2.148 \text{ dB}, \ |\Gamma| = 0.2$

$f = 1 \text{ GHz} \Rightarrow \lambda = \upsilon/f = 30 \times 10^9/10^9 = 30 \text{ cm}$

(a) $D_0 = 2.148 \text{ dB} = 10 \log_{10} D_0(\text{dim}) \Rightarrow D_0(\text{dim}) = 10^{2.148/10} = 1.6398$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0 \cancel{(e_{cd})}^{\,1} (e_r)\ \cancel{\text{PLF}}^{\,1} = \frac{\lambda^2}{4\pi} D_0(\text{dim}) \left[1 - |\Gamma|^2\right]$$

$$A_{em} = \frac{\lambda^2}{4\pi}(1.6398)(1 - |0.2|^2) = \frac{\lambda^2}{4\pi}(1.6398)(1 - 0.04) = \frac{\lambda^2}{4\pi}(1.6396)(0.96)$$

$$A_{em} = \boxed{0.12527\lambda^2}$$

(b) $A_p = ld = \dfrac{\lambda}{2}\left(\dfrac{\lambda}{300}\right) = \dfrac{\lambda^2}{600} = \boxed{0.00167\lambda^2}$

(c) $\varepsilon_{ap} = \dfrac{A_{em}}{A_p} = \dfrac{0.12527\lambda^2}{0.00167\lambda^2} = \boxed{75.162 = 7{,}516.2\%}$

(d) $P_L = W_i A_{em} = 10 \times 10^{-3} \dfrac{\text{W}}{\text{cm}^2}\left(0.12527\lambda^2\right) = 10^{-2}\left[0.12527(30)^2\right]$

$$P_L = 112.743 \times 10^{-2} = 0.112743 \times 10^{+1} \text{ Watts} = 1.12743 \text{ Watts}$$

$$\boxed{P_L = 1.12743 \text{ Watts}}$$

2.78. $W_i = 10^{-3}\ \text{W/m}^2$

$$A_{em} = \frac{\lambda^2}{4\pi} D_0, \ D_0 = 20 \text{ dB} = 10 \log_{10} D_0(\text{dim}) \Rightarrow D_0(\text{dim}) = 100$$

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{10 \times 10^9} = 0.03 \text{ m} = 3 \times 10^{-2} \text{ m}$$

$$A_{em} = \frac{(3 \times 10^{-2})^2}{4\pi} \cdot 100 = \frac{9 \times 10^{-4}}{4\pi} \cdot (100) = 0.716 \times 10^{-2} = 7.16 \times 10^{-3}$$

$$P_{\text{rec}} = 10^{-3} \cdot \left(\frac{9 \times 10^{-2}}{4\pi}\right) = \frac{9 \times 10^{-5}}{4\pi} = 0.716 \times 10^{-5} = 7.16 \times 10^{-6} \text{ Watts}$$

$$P_{\text{rec}} = 7.16 \times 10^{-6} \text{ Watts.}$$

2.79. $f = 10\,\text{GHz}, W^i = 10 \times 10^{-3}\ \text{Watts/cm}^2;\ \text{PLF} = \frac{1}{2} = 0.5 = -3\ \text{dB}$

$D_0 = 12\ \text{dB} = 15.849\ \text{(dimensionless)}; Z_A = 100; Z_c = 50; e_{cd} = 75\% = 0.75$

(a) $A_{em} = \frac{\lambda^2}{4\pi} D_0, \quad \lambda = \frac{30 \times 10^9}{10 \times 10^9} = 3\ \text{cm}$

$$A_{em} = \frac{(3)^2}{4\pi}(15.849) = 11.351$$

$$\boxed{A_{em} = 11.351\ \text{cm}^2}$$

(b) $\Gamma = \frac{Z_A - Z_c}{Z_A + Z_c} = \frac{100 - 50}{100 + 50} = \frac{50}{150} = \frac{1}{3} = 0.3333$

$$e_r = (1 - |\Gamma|^2) = (1 - |0.3333|^2) = 0.88889$$

$$A_{em}(\text{lossy}) = A_{em}(\text{lossless})(e_r)(e_{cd})(\text{PLF})$$

$$= 11.351(0.88889)(0.75)(0.5) = 3.78367$$

$$\boxed{A_{em}(\text{lossy}) = 3.78367\ \text{cm}^2}$$

(c) $P_{\text{receiver}} = W^i A_{em} = 10 \times 10^{-3}(3.78367) = 37.8367 \times 10^{-3}$

$$\boxed{P_{\text{receiver}} = 37.8367 \times 10^{-3}\ \text{Watts}}$$

2.80. $A_p = 10\ \text{cm}^2,\ f = 10\ \text{GHz} \Rightarrow \lambda = 30 \times 10^9/10 \times 10^9 = 3\ \text{cm},\ W^i = 10 \times 10^{-3}\ \text{W/cm}^2$

(a) $A_{em} = \frac{\lambda^2}{4\pi} D_0 = \frac{\lambda^2}{4\pi} G_0 = A_p = 10$

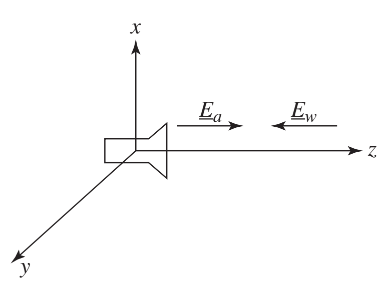
$$\Rightarrow G_0 = \frac{4\pi(10)}{\lambda^2} = \frac{4\pi(10)}{(3)^2} = 13.96 = 11.45\ \text{dB}$$

(b) $P_r = A_{em} W^i(\text{PLF}) = \frac{1}{2}(10)(10 \times 10^{-3}) = 100 \times 10^{-3}/2 = 0.05\ \text{Watts}$

$$P_r = 0.05\ \text{Watts}$$

$$\text{PLF} = \left| \hat{a}_x \cdot \left(\frac{\hat{a}_x + j\hat{a}_y}{\sqrt{2}} \right) \right|^2 = \frac{1}{2}$$

2.81. $\underline{E}_w = (j\hat{a}_x + 2\hat{a}\text{y})e^{+jkz}$
$\underline{E}_a = j\hat{a}_y e^{-jkz}$



(a) $\underline{E}_w = (j\hat{a}_x + 2\hat{a}_y)e^{+jkz} = \underbrace{\frac{(j\hat{a}_x + 2\hat{a}_y)}{\sqrt{5}}}_{\hat{\rho}_w} \sqrt{5} e^{+jkz}$

Elliptical polarization, AR = 2, CCW because:
1. 2 components not of equal magnitude
2. 90° phase difference between the two
3. *x*-component is leading the *y*-component

(b) $\underline{E}_a = \underbrace{j\,\hat{a}_y}_{\hat{\rho}_a}\,e^{-jkz}$

Linear polarization, AR = ∞, no rotation because one component.

(c) $A_{em} = \dfrac{\lambda^2}{4\pi}D_0,\ D_0 = 2.15\,\text{dB} = 10\log_{10}D_0(\text{dimensionless})$

$D_0(\text{dimensionless}) = 10^{2.15/10} = 10^{0.215} = 1.641$

$D_0(\text{dimensionless}) = 1.641$

$$A_{em} = \frac{\lambda^2}{4\pi}D_0 = \frac{\lambda^2}{4\pi}(1.641) = \boxed{0.131\lambda^2 = 0.410\pi\lambda^2}$$

(d) $A_{em} = \dfrac{\lambda^2}{4\pi}D_0 = (1-|\Gamma|^2)\text{PLF} = \dfrac{\lambda^2}{4\pi}(1.641)(1-|0.5|^2)\left|\dfrac{(j\hat{a}_x + 2\hat{a}_y)}{\sqrt{5}}\cdot(\hat{a}_y)\right|^2$

$$A_{em} = 0.131\lambda^2(1-0.25)\left(\frac{2}{\sqrt{5}}\right)^2 = 0.131\lambda^2(0.75)\left(\frac{4}{5}\right) = 0.079\lambda^2$$

(e) $P_L = A_{em}W_i = 0.079\lambda^2(10\times10^{-3}/\lambda^2) = 0.79\times10^{-3}$ Watts

$$P_L = 0.79\times10^{-3}\ \text{Watts} = 0.79\ \text{mWatts}$$

2.82. $\underline{W}_{\text{rad}} = \underline{W}_{\text{ave}} \simeq C_0\dfrac{1}{r^2}\cos^4(\theta)\hat{a}_r \qquad (0\le\theta\le\pi/2, 0\le\phi\le2\pi)$

(a)
$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi/2}\underline{W}_{\text{rad}}\cdot d\underline{s} = \int_0^{2\pi}\int_0^{\pi/2}\hat{a}_r W_{\text{rad}}\cdot\hat{a}_r r^2\sin\theta\,d\theta\,d\phi$$
$$= C_0\int_0^{2\pi}\int_0^{\pi/2}\cos^4\theta\sin\theta\,d\theta\,d\phi = 2\pi C_0\int_0^{\pi/2}\cos^4\theta\sin\theta\,d\theta$$
$$= 2\pi C_0\left(-\frac{\cos^5\theta}{5}\right)_0^{\pi/2}$$
$$P_{\text{rad}} = 2\pi C_0\left(0+\frac{1}{5}\right) = \frac{2\pi}{5}C_0 = 1.2566C_0$$

(b) $D_0 = \dfrac{4\pi U_{\max}}{P_{\text{rad}}} \Rightarrow U_{\max} = r^2\,W_{\text{rad}}|_{\max} = C_0\cos^4\theta|_{\max} = C_0$

$$D_0 = \frac{4\pi C_0}{2\pi C_0/5} = 10 = 10\log_{10}(10) = 10\ \text{dB}$$

(c) $D_0 = 10$ toward $\theta = 0°$

(d) $A_{em} = \dfrac{\lambda^2}{4\pi}D_0 \quad \lambda = \dfrac{c}{f} = \dfrac{3\times10^8\ \text{m/sec}}{1\times10^9} = 0.3\ \text{m}$

$$A_{em} = \frac{(0.3)^2}{4\pi}(10) = \frac{0.09}{4\pi}(10) = \frac{0.225}{\pi} = 0.0716\ \text{m}^2$$

(e) $P_L = A_{em}W^i = 0.0716\times(10\times10^{-3}) = 0.716\times10^{-3}$ Watts

2.83. $D_0(\lambda/2) = 2.286 \text{ dB} = 10^{0.2286} = 1.69278 \text{ (dim)}$

$D_0(\lambda/4) = 5.286 \text{ dB} = 10^{0.5286} = 3.37754 \text{ (dim)}$

$$P_{\text{rad}} = 10 \text{ watts},\ f = 1,900 \text{ MHz} \Rightarrow \lambda = \frac{30\times 10^9}{1.9\times 10^9} = 15.78947 \text{ cm}$$

(a) $$W_{\text{rad}}(\text{isotropic}) = \frac{P_{\text{rad}}}{4\pi r^2} = \frac{10}{4\pi(1,000\times 100)^2} = \frac{10}{4\pi\times 10^{10}} = 0.07958\times 10^{-9}$$
$$= 79.58\times 10^{-12} \text{ W/cm}^2$$

$$W_{\text{rad}}(\lambda/2) = W_{\text{rad}}(\text{isotropic})D_0 = 79.58\times 10^{-12}(1.69278) = 134.711\times 10^{-12}$$

$$W_{\text{rad}} = (\lambda/2) = \boxed{134.711\times 10^{-12} \text{ W/cm}^2}$$

(b) $D_0(\lambda/4) = 5.286 \text{ dB} = 3.37754 \text{ dim.}$

$$A_{em} = \frac{\lambda^2}{4\pi}D_0 = \frac{(15.78947)^2}{4\pi}(3.37754) = \boxed{67 \text{ cm}^2}$$

(c) $P_{\text{rec}} = W_{\text{rad}}(\lambda/2)A_{em}(\lambda/4) = 134.711\times 10^{-12}(67) = 9{,}025.637\times 10^{-12}$

$$P_{\text{rec}} = \boxed{9.0256\times 10^{-9} \text{ Watts}}$$

2.84. $A_{em} = \frac{\lambda^2}{4\pi}e_t D_0 = \frac{\lambda^2}{4\pi}G_0$

(a) $G_0 = 14.8 \text{ dB} \Rightarrow G_0(\text{power ratio}) = 10^{1.48} = 30.2$

$f = 8.2 \text{ GHZ} \Rightarrow \lambda = 3.6585 \text{ cm}$

$$A_{em} = \frac{(3.6585)^2}{4\pi}(30.2) = 32.167 \text{ cm}^2$$

The physical aperture is equal to $A_p = 5.5(7.4) = 40.7 \text{ cm}^2$

(b) $G_0 = 16.5 \text{ dB} \Rightarrow G_0(\text{power ratio}) = 10^{1.65} = 44.668$

$f = 10.3 \text{ GHz} \Rightarrow \lambda = 2.912 \text{ cm}$

$$A_{em} = \frac{(2.912)^2}{4\pi}(44.668) = 30.142 \text{ cm}^2$$

(c) $G_0 = 18.0 \text{ dB} \Rightarrow G_0(\text{power ratio}) = 10^{1.8} = 63.096$

$f = 12.4 \text{ GHz} \Rightarrow \lambda = 2.419 \text{ cm}$

$$A_{em} = \frac{(2.419)^2}{4\pi}(63.096) = 29.389 \text{ cm}^2$$

2.85. $P_{in} = 100 \text{ Watts};\ Z_c = 75 \text{ ohms};\ Z_{in} = Z_A = 100;\ e_{cd} = 50\%$

$$U(\theta,\phi) = B_0\sin\theta;\quad 0\le\theta\le 180^\circ,\ 0\le\phi\le 360^\circ$$

(a) $$\Gamma = \frac{Z_A - Z_c}{Z_A + Z_c} = \frac{100-75}{100+75} = \frac{25}{175} = \frac{1}{7} = 0.14286$$

$e_r = (1-|\Gamma|^2) = (1-|0.1428|^2) = (1-0.0204) = 0.9796 = 97.96\%$

$$\boxed{e_r = 0.9796 = 97.96\%}$$

(b) $e_0 = e_{cd}e_r = 0.5(0.9796) = \boxed{0.4898 = 48.98\%}$

(c) $$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U \sin\theta \, d\theta \, d\phi = B_0 \int_0^{2\pi}\int_0^{\pi} \sin^2\theta \, d\theta \, d\phi = 2\pi B_0 \int_0^{\pi} \sin^2\theta \, d\theta$$

$$P_{\text{rad}} = 2\pi B_0 \int_0^{\pi} \left[\frac{1-\cos(2\theta)}{2}\right] d\theta = \pi B_0 \left[\theta - \frac{1}{2}\sin(2\theta)\right]_0^{\pi} = \pi^2 B_0$$

$$P_{\text{rad}} = P_{in}(e_r e_{cd}) = P_{in}(e_0) = 100(0.4898) = 48.98 \text{ watts}$$

$$48.98 = \pi^2 B_0 \Rightarrow \boxed{B_0 = \left(\frac{48.98}{\pi^2}\right) = 4.9627}$$

(d) $$U = B_0 \sin\theta = 4.9627 \ \sin\theta \Rightarrow U_{\max} = 4.9627$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(4.9627)}{48.98} = 1.2732$$

$$\boxed{D_0 = 1.2732 = 1.049 \text{ dB}}$$

(e) $$W_{\text{rad}|\max} = \frac{U_{\max}}{r^2} = \frac{4.9627}{[1{,}000(100)]^2} = \frac{4.9627}{(10^5)^2} = 4.9627 \times 10^{-10}$$

$$\boxed{W_{\text{rad}|\max} = 0.49627 \times 10^{-9} \text{ Watts/cm}^2}$$

2.86. $A_{em} = \frac{\lambda^2}{4\pi} D_0$

From Problem 2.61:
Computer Program **Directivity:** $D_0 = 80.2511 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(80.2511) = 6.386\lambda^2$
Table 12.1: $D_0 = 75.398 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(75.398) = 6\lambda^2$

2.87. $A_{em} = \frac{\lambda^2}{4\pi} D_0$

From Problem 2.62:
Computer Program **Directivity:** $D_0 = 62.4635 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(62.4635) = 4.971\lambda^2$
Table 12.1: $D_0 = 61.072 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(61.072) = 4.86\lambda^2$

2.88. $A_{em} = \frac{\lambda^2}{4\pi} D_0$

From Problem 2.63:
Computer Program **Directivity:** $D_0 = 4.2443 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(4.2443) = 0.3378\lambda^2$
Table 12.1: $D_0 = 2.627 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(2.627) = 0.20905\lambda^2$

2.89. $A_{em} = \frac{\lambda^2}{4\pi} D_0$

From Problem 2.64:
Computer Program **Directivity:** $D_0 = 92.947 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(92.947) = 7.396\lambda^2$
Table 12.2: $D_0 = 88.826 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(88.826) = 7.068\lambda^2$

2.90. $A_{em} = \frac{\lambda^2}{4\pi} D_0$

From Problem 2.65:
Computer Program **Directivity:** $D_0 = 75.1735 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(75.1735) = 5.982\lambda^2$
Table 12.2: $D_0 = 74.2589 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(74.2589) = 5.909\lambda^2$

2.91. $A_{em} = \dfrac{\lambda^2}{4\pi} D_0$

From Problem 2.66:
Computer Program **Directivity:** $D_0 = 8.0236 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(8.0236) = 0.638\lambda^2$
Table 12.2: $D_0 = 4.791 \Rightarrow A_{em} = \frac{\lambda^2}{4\pi}(4.791) = 0.3813\lambda^2$

2.92. Gain = 30 dB, $f = 2$ GHZ, $P_{\text{rad}} = 5$ W
Receiving antenna VSWR = 2, efficiency = 95%

$$\underline{E}_R = (2\hat{a}_x + j\hat{a}_y)F_R(\theta,\phi), \text{ Use Friis transmission formula (2.118)}$$

$$P_r = P_t e_{cdt} e_{cdr}(1-|\Gamma_t|^2)(1-|\Gamma_r|^2)\left(\frac{\lambda}{4\pi R}\right)^2 D_t(\theta_t,\phi_t) D_r(\theta_r,\phi_r)\cdot \text{PLF}$$

$$P_r = 10^{-14}\ \text{W}, e_{cdt} = 1 \text{ (we assume that)},\ e_{cdr} = 0.95, 1-|\Gamma_r|^2 = 1$$

$$\text{Since VSWR} = 2 \Rightarrow |\Gamma_r| = \left|\frac{\text{VSWR}-1}{\text{VSWR}+1}\right| = \frac{2-1}{2+1} = \frac{1}{3},\ (1-|\Gamma_r|^2) = 8/9$$

$$\lambda = \frac{3\times 10^8}{2\times 10^9} = 0.15 \text{ m},\ R = 4000\times 10^3 \text{ m},$$

$$\text{Hence } \left(\frac{\lambda}{4\pi R}\right)^2 = \left(\frac{0.15}{4\pi 4000\times 10^3}\right)^2 = 8.9\times 10^{-18}$$

$$D_t = 30\ \text{dB} = 10^3, \text{PLF} \Rightarrow \begin{cases} \hat{\rho}_t = \frac{1}{\sqrt{2}}(\hat{a}_x + j\hat{a}_y) \Rightarrow |\hat{\rho}_t\cdot\hat{\rho}_r|^2 = 0.1 \\ \hat{\rho}_r = \frac{1}{\sqrt{5}}(2\hat{a}_x + j\hat{a}_y) \end{cases}$$

$$\Rightarrow 10^{-14} = 5(1)(0.95)(1)\left(\frac{8}{9}\right)(8.9\times 10^{-18})(10^3)D_r(0.1)$$

$$D_r = 2.661$$

$$\text{Hence } A_{em} = \frac{\lambda^2}{4\pi}2.661 = 0.00476 \text{ m}^2$$

2.93. $U(\theta,\phi) = \left\{\begin{array}{ll} \cos^4\theta, & 0^\circ \le \theta \le 90^\circ \\ 0, & 90^\circ \le \theta \le 180^\circ \end{array}\right\} 0^\circ \le \phi \le 360^\circ$

$$A_{em} = \frac{\lambda^2}{4\pi}D_0$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}}$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U(\theta,\phi)\sin\theta\, d\theta\, d\phi = 2\pi\int_0^{\pi/2}\cos^4(\theta)\sin\theta\, d\theta = 2\pi\left[-\frac{\cos^5\theta}{5}\right]_0^{\pi/2}$$

$$P_{\text{rad}} = 2\pi\left(-0+\frac{1}{5}\right) = \frac{2\pi}{5}$$

$$D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{4\pi(1)}{2\pi/5} = 10$$

$$A_{em} = \frac{\lambda^2}{4\pi}D_0 = \frac{\lambda^2}{4\pi}(10) = \frac{10\lambda^2}{4\pi},\ \lambda = \frac{3\times10^8}{10^{10}} = 3\times10^{-2} = 0.03\,\text{m}$$

$$A_{em} = \frac{10(0.03)^2}{4\pi} = \frac{10(3\times10^{-2})^2}{4\pi} = \frac{10(9\times10^{-4})}{4\pi} = 7.16197\times10^{-4}$$

$$A_{em} = 7.16197\times10^{-4}$$

2.94. 1 status mile = 1609.3 meters, 22,300(status miles) = 3.588739×10^7 m

(a) $P_i = \dfrac{P_{\text{rad}}}{4\pi R^2} = \dfrac{8\times10^{-14}}{4\pi\times(3.58874)} = 4.943\times10^{-16}\,\text{Watts/m}^2.$

(b) $A_{em} = \dfrac{\lambda^2}{4\pi}D_0, \quad (D_0 = 60\,\text{dB} = 10^6), \quad (\lambda = 0.15\,\text{m})$

$$A_{em} = \frac{(0.15)^2}{4\pi}(10^6) = 1790.493\,\text{m}^2$$

$$P_{\text{received}} = A_{em}\cdot P_i = (1790.493)(4.943\times10^{-16})$$
$$= 8.85\times10^{-13}\,\text{Watts.}$$

2.95. $A_{em} = 0.7162\,\text{m}^2$

$$A_{em} = \left(\frac{\lambda}{4\pi}\right)^2 e_{cd}(1-|\Gamma|^2)|\hat{\rho}_w\cdot\hat{\rho}_a|^2 D_0$$

$$D_0 = \frac{A_{em}}{\left(\dfrac{\lambda}{4\pi}\right)^2(1-1\Gamma|^2)},\ \Gamma = \frac{75-50}{75+50} = 0.2,\ \lambda = \frac{3\times10^8}{100\times10^6} = 3\,\text{m}$$

$$D_0 = \frac{0.7162}{\dfrac{3^2}{4\pi}(1-|0.2|^2)}$$

$$D_0 = 1.0417$$

2.96. $P_r = W_iA_{em} = W_ie_{cd}(1-|\Gamma|^2)\left(\dfrac{\lambda^2}{4\pi}\right)^2 D_0|\hat{\rho}_w\cdot\hat{\rho}_a|^2$

$$W_i = 5\,\text{W/m}^2, e_{cd} = 1(\text{lossless}),\ \Gamma = \frac{Z_{in}-Z_0}{Z_{in}+Z_0} = \frac{73-50}{73+50} = 0.187$$

$$\lambda = \frac{3\times10^8}{10\times10^6} = 30\,\text{m},\ D_0 = 2.156\,\text{dB} = 1.643, \text{PLF} = 1$$

$$P_r = (5)(1)[1-(0.187)^2]\left(\frac{30^2}{4\pi}\right)(1.643)(1) = 567.78\,\text{Watts}$$

$$P_r = 567.78\,\text{Watts.}$$

2.97. $\dfrac{P_r}{P_t} = \left(\dfrac{\lambda}{4\pi R}\right)^2 G_{0r}G_{0t}, G_{0r} = G_{0t} = 16.3 \Rightarrow G_0(\text{power ratio}) = 42.66$

$$f = 10\,\text{GHz} \Rightarrow \lambda = 0.03\,\text{meters.}$$

$$\text{VSWR} = 1.1 \Rightarrow |\Gamma| = \frac{\text{VSWR}-1}{\text{VSWR}+1} = \frac{1.1-1}{1.1+1} = \frac{0.1}{2.1} = 0.0476$$

$$P_t = 200\,\text{m watts} = 0.2\,\text{Watts}$$

(a) $R = 5\,\text{m}: P_r = \left[\dfrac{0.03}{4\pi(5)}\right]^2 (42.66)^2(0.2)[1 - |\Gamma|^2]^2$

$$= 82.9[1 - (0.0476)^2]^2 = 82.9(0.9977)^2 = 82.5$$

(b) $R = 50\,\text{m}: \; P_r = 0.825\,\mu\text{Watts}$

(c) $R = 500\,\text{m}: \; P_r = 8.25\,n\text{Watts}$

2.98. $\dfrac{p_r}{p_t} = |\hat{\rho}_t \cdot \hat{\rho}_r|^2 \left(\dfrac{\lambda}{4\pi R}\right)^2 G_{0t}G_{0r}$

$G_{0t} = 20\text{ dB} \Rightarrow G_{0t}(\text{power ratio}) = 10^2 = 100$

$G_{0r} = 15\text{ dB} \Rightarrow G_{0r}(\text{power ratio}) = 10^{1.5} = 31.623$

$f = 1\text{ GHz} \Rightarrow \lambda = 0.3\text{ meters}$

$R = 1 \times 10^3\text{ meters}$

(a) For $|\hat{\rho}_t \cdot \hat{\rho}_r|^2 = 1$

$$P_r = \left(\frac{0.3}{4\pi \times 10^3}\right)^2 (100)(31.623)(150 \times 10^{-3}) = 270.344\,\mu\text{Watts}$$

(b) When transmitting antennas is circularly polarized and receiving antenna is linearly polarized, the PLF is equal to

$$|\hat{\rho}_t \cdot \hat{\rho}_r|^2 = \left|\left(\frac{\hat{a}_x \pm j\hat{a}_y}{\sqrt{2}}\right) \cdot \hat{a}_x\right|^2 = \frac{1}{2}$$

Thus

$$P_r = \frac{1}{2}(270.344 \times 10^{-6}) = 135.172 \times 10^{-6} = 135.172\,\mu\text{Watts}$$

2.99. Lossless: $e_{cd} = 1$, polarization matched: $|\hat{\rho}_w \cdot \hat{\rho}_a|^2 = 1$, line matched: $(1 - |\Gamma|^2) = 1$

$D_0 = 20\text{ dB} = 10^2 = 100 = D_{0r} = D_{0t}$

$P_r = P_t \left(\dfrac{\lambda}{4\pi R}\right)^2 D_{0t}D_{0r} = 10 \left(\dfrac{\lambda}{4\pi \cdot 50\lambda}\right)^2 (100)(100) = 0.253\text{ Watts}$

$P_r = 0.253\text{ Watts}$

2.100. Lossless: $e_{cd} = 1$, PLF = 1. Line matched: $(1 - |\Gamma|^2) = 1$.
$D_0 = 30\,\text{dB} = 10^3 = 1000 = D_{0r} = D_{0t}$

$P_r = P_t \left(\dfrac{\lambda}{4\pi \cdot 100\lambda}\right)^2 (1000)^2 = 20 \left(\dfrac{1}{4\pi}\right)^2 100 = 12.665\text{ Watts}$

2.101. $G_{0r} = 20\,\text{dB} = 100, G_{0t} = 25\,\text{dB} = 316.23,\; \lambda = \dfrac{3 \times 10^8}{3 \times 10^9} = 0.1\text{ m}$

$$P_r = P_t|\hat{\rho}_t \cdot \hat{\rho}_r|^2 \left(\frac{\lambda}{4\pi R}\right)^2 G_{0r}G_{0t}$$

$$= 100(1) \left(\frac{0.1}{4\pi \times 500}\right)^2 (100)(316.23)$$

$P_r = 8 \times 10^{-4}\text{ Watts}$

2.102. $f = 10\text{ GHz} \rightarrow \lambda = \dfrac{3\times 10^8}{10^{10}} = 0.03\text{ m}$

$G_{0t} = G_{0r} = 15\text{ dB} = 10^{1.5} = 31.62$ (dimensionless)

$R = 10\text{ km} = 10^4\text{ m}$

$P_r \geq 10\text{ nW} = 10^{-8}\text{ W}$

$|\hat{\rho}_t \cdot \hat{\rho}_r|^2 = -3\text{ dB} = \dfrac{1}{2}$

Friis Transmission Equation:

$$\frac{P_r}{P_t} = G_{0t}G_{0r}\left(\frac{\lambda}{4\pi R}\right)^2 |\hat{\rho}_t \cdot \hat{\rho}_r|^2$$

$$= (10^{1.5})^2 \left(\frac{0.03}{4\pi \times 10^4}\right)^2 \left(\frac{1}{2}\right) = 2.85 \times 10^{-11}$$

$$P_t = \frac{P_r}{2.85\times 10^{-11}}$$

$$P_r \geq 10^{-8}\text{ W} \rightarrow (P_t)_{\min} = 351\text{ W}$$

2.103. $$\frac{P_r}{P_t} = (\text{PLF})e_t e_r D_{0t} D_{0r}\left(\frac{\lambda}{4\pi R}\right)^2$$

$$= (\text{PLF})(e_{rt}e_{cdt})(e_{rr}e_{cdr})\left(\frac{\lambda}{4\pi R}\right)^2 D_{0t}D_{0r}$$

$$\frac{P_r}{P_t} = (1)[e_{rt}(1)][e_{rr}(1)]\left(\frac{\lambda}{4\pi R}\right)^2 D_{0t}D_{0r}$$

$$\lambda = \frac{c}{f} = \frac{3\times 10^8}{10^8} = 3\text{ m}, R = 10\times 10^3 = 10^4$$

$$\left(\frac{\lambda}{4\pi R}\right)^2 = \left(\frac{3}{4\pi\times 10^4}\right)^2 = \left(\frac{3}{4\pi}\times 10^{-4}\right)^2$$

$$= (0.2387\times 10^{-4})^2 = 5.699\times 10^{-2}\times 10^{-8}$$

$$\left(\frac{\lambda}{4\pi R}\right)^2 = 5.699\times 10^{-10}$$

$$e_{rt} = e_{rr} = (1-|\Gamma|^2) = \left(1 - \left|\frac{73.3-50}{73.3+50}\right|^2\right) = \left(1 - \left|\frac{23.3}{12.3}\right|^2\right)$$

$$= [1-(0.18897)^2] = (1-0.0357) = 0.9643$$

$$e_{cdt} = e_{cdr} = 1$$

$$D_{0t} = D_{0r} = 1.643$$

$$\frac{P_r}{P_t} = (0.9643)^2(1.643)^2(5.699\times 10^{-10})$$

$$= (0.92987)(2.699)(5.699\times 10^{-10})$$

$$= 2.51(5.699\times 10^{-10}) = 14.305\times 10^{-10}$$

$$P_t = \frac{P_r}{14.305 \times 10^{-10}} = 6.99 \times 10^{-2} \times 10^{10}(1 \times 10^{-6})$$
$$= 6.99 \times 10^2$$
$$P_t = 699 \text{ Watts}$$

2.104. $\frac{P_r}{P_t} = \left(\frac{\lambda}{4\pi R}\right)^2 G_{0t}G_{0r},\ \lambda = \frac{3 \times 10^8}{9 \times 10^9} = \frac{3 \times 10^8}{90 \times 10^8} = \frac{1}{30}$

$$R = 10{,}000 \text{ meter} = \frac{10{,}000}{1/30}\lambda = 3 \times 10^5\lambda$$
$$\frac{P_r}{P_t} = \left[\frac{\lambda}{4\pi(3 \times 10^5\lambda)}\right]^2 G_0^2 = \frac{10 \times 10^{-6}}{10} = 10^{-6}$$
$${G_0}^2 = 10^{-6}(4\pi \times 3 \times 10^5)^2$$
$$G_0 = 10^{-3}(4\pi \times 3 \times 10^5) = 12\pi \times 10^2 = 1200\pi$$
$$G_0 = 1200\pi = 3{,}769.91 = 10\log_{10}(3{,}769.91) \text{ dB}$$
$$G_0 = 3{,}769.91 = 35.76 \text{ dB}$$

2.105. $R = 16 \times 10^3$ m, $f = 2$ GHz, $G_{0t} = 20$ dB, $P_t = 100$ watts,

$P_r = 5 \times 10^{-9}$ Watts $\Rightarrow G_{0r} = ?$

$$G_{0t} = 20 \text{ dB} = 10\log_{10}[G_{0t}(\text{dim})] \Rightarrow G_{0t}(\text{dimensionless}) = 10^2 = 100$$
$$G_{0t}(\text{dimensionless}) = 100$$
$$f = 2 \text{ GHz} \Rightarrow \lambda = \frac{3 \times 10^8}{2 \times 10^9} = 0.15 \text{ meters}$$

Friis Transmission Equation (2-119):

$$\frac{P_r}{P_t} = G_{0t}G_{0r}\left(\frac{\lambda}{4\pi R}\right)^2 \text{PLF} \Rightarrow G_{0r} = \frac{P_r}{P_t}\left(\frac{1}{G_{0t}}\right)\left(\frac{4\pi R}{\lambda}\right)^2\left(\frac{1}{\text{PLF}}\right)$$
$$G_{0r} = \frac{5 \times 10^{-9}}{100}\left(\frac{1}{100}\right)\left[\frac{4\pi(16 \times 10^3)}{0.15}\right]^2\left(\frac{2}{1}\right)$$
$$= \frac{10 \times 10^{-9} \times 10^6}{10^4}\left[\frac{4\pi(16)}{0.15}\right]^2 = 10^{-6}(1,340.413)^2$$
$$G_{0r} = 1,796,706.65 \times 10^{-6} = 1.7967 = 2.545 \text{ dB}$$
$$\boxed{G_{0r} = 1.7967 = 2.545 \text{ dB}}$$

2.106. $\sigma = \pi \text{a}^2 = 25\pi\lambda^2$

$$G_{0t} = G_{0r} = 16.3 \text{ dB} \Rightarrow G_{0t} \text{ (power ratio)} = 10^{1.63} = 42.66$$
$$f = 10 \text{ GHz} \Rightarrow \lambda = 0.03 \text{ m}$$
$$\frac{P_r}{P_t} = \sigma\frac{G_{0t}G_{0r}}{4\pi}\left(\frac{\lambda}{4\pi R_1 R_2}\right)^2$$

(a) $R_1 = R_2 = 200\lambda = 6$ meters:

$$P_r = 25(\pi\lambda^2)\frac{(42.66)^2}{4\pi}\left[\frac{\lambda}{4\pi(200\lambda)^2}\right]^2(0.2) = 9.00\ n\text{Watts}$$

(b) $R_1 = R_2 = 500\lambda = 15$ meters:

$P_r = 0.23\ n$Watts

2.107. $P_r = P_t\sigma\frac{G_{0t}G_{0r}}{4\pi}\left[\frac{\lambda}{4\pi R_1 R_2}\right]^2, \quad \lambda = \frac{3\times10^8}{5\times10^9} = 0.06$ m

$$P_r = 10^5(3)\frac{150^2}{4\pi}\left[\frac{0.06}{4\pi(10^6)}\right]^2$$

$P_r = 1.22\times10^{-8}$ Watts

2.108. $\frac{P_r}{P_t} = \sigma\frac{G_{0r}G_{0t}}{4\pi}\left[\frac{\lambda}{4\pi R_1R_2}\right]^2 \Rightarrow \sigma = \frac{P_r(4\pi)}{P_tG_{0r}G_{0t}}\left[\frac{4\pi R_1R_2}{\lambda}\right]^2$

$$\lambda = \frac{3\times10^8}{10\times10^9} = 0.03 \text{ m}$$

$$\therefore \sigma = \frac{10^{-16}(4\pi)}{1000(80)(80)}\left[\frac{4\pi(10^4)(10^4)}{0.03}\right]^2 = 3.445 \text{ m}^2$$

2.109. $\sigma = \frac{P_r4\pi}{P_tG_{0r}G_{0t}}\left[\frac{4\pi R_1R_2}{\lambda}\right]^2$

$$\lambda = \frac{3\times10^8}{3\times10^9} = 0.1 \text{ m}$$

$$\sigma = \frac{10^{-16}(4\pi)}{100(80)(80)}\left[\frac{4\pi(10^4)(10^4)}{0.1}\right]^2 = 0.31 \text{ m}^2$$

2.110. $\sigma = 0.85\lambda^2$

$$\frac{P_r}{P_t} = \sigma\frac{G_{0t}G_{0r}}{4\pi}\left(\frac{\lambda}{4\pi R_1R_2}\right)^2|\hat{\rho}_w\cdot\hat{\rho}_r|^2$$

$\sigma = 0.85\lambda^2,\ G_{0t} = G_{0r} = 15$ dB $\Rightarrow G_{0t} = G_{0r} = 31.6228$ (dimensionless)

$R_1 = R_2 = 100$ meter $\Rightarrow R_1 = R_2 = 1,000\lambda$

$$f = 3 \text{ GHz} \Rightarrow \lambda = \frac{3\times10^8}{3\times10^9} = 0.1 \text{ meters}$$

$$|\hat{\rho}_w\cdot\hat{\rho}_r|^2 = 1 \text{ dB} \Rightarrow |\hat{\rho}_w\cdot\hat{\rho}_r|^2 = 0.7943$$

$$\frac{P_r}{P_t} = 0.85\lambda^2\frac{(31.6228)^2}{4\pi}\left(\frac{\lambda}{4\pi\times10^6\lambda^2}\right)^2(0.7943)$$

$$= \frac{0.85(31.6228)^2(0.7943)}{(4\pi)^3(10^{12})} = 0.3402\times10^{-12}$$

$P_r = 0.3402\times10^{-12}(10^2) = 0.3402\times10^{-10} = 34.02\times10^{-12}$ Watts

$P_r = 34.02\ p$Watts

2.111. $T_a = T_A e^{-2\alpha l} + T_0(1 - e^{-2\alpha l})$

$T_A = 5$ K

$T_0 = 72°\text{F} = \frac{5}{9}(72 - 32) + 273 = 295.2$ K

$-4 \text{ dB} = 20 \log_{10} e^{-\alpha} = -\alpha(20)\log_{10} e = -\alpha(20)(0.434)$

$\alpha = \frac{4}{8.68} = 0.460$ Nepers/100 ft $= 0.0046$ Nepers/ft.

(a) $l = 2$ feet:

$$T_a = 5e^{-2(0.0046)^2} + 295.2[1 - e^{-2(0.0046)^2}] = 4.91 + 5.38 = 10.29 \text{ K}$$

(b) $l = 100$ feet;

$$T_a = 5e^{-2(0.0046)100} + 295.2[1 - e^{-2(0.0046)100}] = 179.72 \text{ K}$$

2.112. $T_a = T_A e^{-\int_0^d 2\alpha(z)\,dz} + \int_0^d \varepsilon(z) T_m(z) e^{-\int_z^d 2\alpha(z')\,dz'}\,dz$

If $\alpha(z) = \alpha_0 =$ Constant

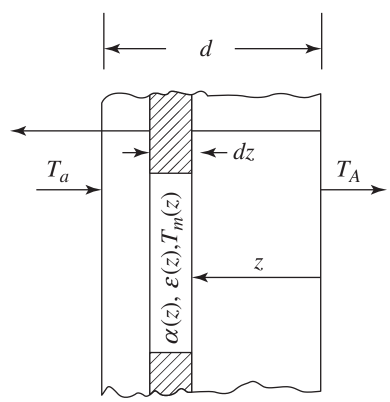
$$T_a = T_A e^{-2\alpha_0 d} + \int_0^d \varepsilon(z) T_m(z) e^{-2\alpha_0(d-z)}\,dz$$

$$T_a = T_A e^{-2\alpha_0 d} + e^{-2\alpha_0 d} + \int_0^d \varepsilon(z) T_m(z) e^{+2\alpha_0 z}\,dz$$

If $T_m(z) = T_0 =$ Constant and $\varepsilon(z) = \varepsilon_0 =$ constant

$$T_a = T_A e^{-2\alpha_0 d} + \varepsilon_0 T_0 e^{-2\alpha_0 d} \int_0^d e^{2\alpha_0 z}\,dz$$

$$T_a = T_A e^{-2\alpha_0 d} + \frac{\varepsilon_0}{2\alpha_0} T_0 e^{-2\alpha_0 d}(e^{2\alpha_0 d} - 1)$$



For $\varepsilon_0 = 2\alpha_0$:

$$T_a = T_A e^{-2\alpha_0 d} + T_0 e^{-2\alpha_0 d}(e^{2\alpha_0 d} - 1)$$
$$= T_A e^{-2\alpha_0 d} + T_0(1 - e^{-2\alpha_0 d})$$